1. Prove that if we select 151 integers from the set A = {1, 2, 3, …, 300}, there exist two integers m and n in the selection where gcd(m, n) = 1. (10%)

2. The Fibonacci sequence is defined recursively:
   $F_0 = 0; \quad F_1 = 1;$
   $F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$
   Prove that $\sum_{i=1}^{2n} F_i F_{i-1} = F_{2n}^2$, for positive integer *n*. (15%)

3. Three integers are selected from the integers 1, 2, …, 99, 100. In how many ways can these integers be selected such that their sum is divisible by 4? (10%)

4. Let *I* = {0, 1}, *O* = {0, 1}, and $S = \{s_{0,}\ s_1, s_2, s_3\}$, where *I* is the input symbol set, *O* the output symbol set, and *S* the state set. Construct a state diagram for a finite state machine that recognizes each occurrence of 1010 in a string $x \in I^*$. Overlapping is allowed. For example, for an input string *x* = 01010101101011, the output string = 00001010000100. (15%).

5. If A is n x n nonsingular matrix, show the follows. (10%)
   (1) $(\text{adj A})^{-1} = \det(A^{-1})$ A. (5%)
   (2) $\det(A^{-1})$ A = adj (A^{-1}). (5%)

6. Given the matrix A as follows. (20%)

$$\begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 15 & 8 \end{pmatrix}$$

   (1) Find the basis of row space of matrix A. (5%)
   (2) Find the basis of column space of matrix A. (5%)
   (3) Find the basis of null space of matrix A. (5%)
   (4) What are rank(A) and N(A)? (5%)

7. Given the matrix A as follows. (20%)

$$\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}$$

   (1) Find the eigenvalues of matrix A. (5%)
   (2) Find the eigenvectors of matrix A. (5%)
   (3) Let D be a diagonal matrix, and $D = X^{-1}AX$, what are matrices D and X? (5%)
   (4) Let $A = B^4$, what is the matrix B? (5%)