# 行政院國家科學委員會專題研究計畫 期末報告

## 在邊界積分方程的新內場法研究

計 畫 類 別 ：個別型
計 畫 編 號 ：NSC 101-2115-M-216-001-
執 行 期 間 ：101 年 08 月 01 日至 102 年 07 月 31 日
執 行 單 位 ：中華大學應用統計學系

計畫主持人：李明恭

計畫參與人員：碩士班研究生-兼任助理人員：蔡其璋
碩士班研究生-兼任助理人員：洪挺軒
碩士班研究生-兼任助理人員：高士傑
碩士班研究生-兼任助理人員：莊美惠

報 告 附 件 ：出席國際會議研究心得報告及發表論文

公 開 資 訊 ：本計畫可公開查詢

中 華 民 國 102 年 10 月 16 日

中 文 摘 要 ： 教育部工程科學學術獎章得主海洋大學陳正宗特聘教授，近年來和他的研究團隊發展了一個邊界積分方程的數值方法稱為零場法。該方法是將基本解以退化核方式將基本解用在邊界元素法產生之格林公式內。已知或未知的圓形邊界值以富立葉級數展開，利用正交性，一個代數方程式很容易產生。零場法已經成功運用到許多橢圓偏微分方程及應用問題，在本計畫中我們將針對拉普拉斯方程探討。我們這個提案的基礎是一篇將在 EABE 發表的論文[Li, Huang, Laiw, Lee*, 2012]，針對零場法在邊界佈點的合理性已經證明是可行的施行方式，而且對於病態性的情形作了理論證明，我們也發現在邊界或是場域裡佈點不會影響太多數值解的準確性。

所以在計畫中，我們提出一個新的方法稱為內場法，它是零場法將場點佈在邊界上的一個特殊型態。另外也提出一個符合物理性質的 flux 守恆條件的保守演算法，可以將變數更進一步減少。內場法比起零場法更簡單，只需一個方程式同時處理內邊界及外邊界，不像零場法需要兩個場域方程式。所以我們將提出有 flux 守恆及沒有 flux 守恆條件的內場法，一個具圓洞的圓型場域模型會用在數值計算上，我們將以條件係數及有效條件係數比較、探討零場法及內場法計算的數值穩定性，部分收斂性分析也是計畫的重點。

中文關鍵詞： 邊界積分方程、基本解、富立葉級數、橢圓偏微分方程、拉普拉斯方程、 零場法、內場法、條件係數

英 文 摘 要 ： Recently, the null field method (NFM) is proposed by the 2011 Ministry of Education Academic Medal Winner in engineering science, Prof. J. T. Chen with his research groups. In NFM, the fundamental solutions with the source nodes outside of the solution domain are used in the Green formulas by the boundary integral equations from boundary element methods. The Fourier expansions of the known and the unknown boundary conditions on the circular boundaries are chosen, so that the explicit discrete matrices are easily constructed by means of orthogonality of Fourier functions. The NFM has been applied to elliptic problems in circular domains with multiple circular holes, and many engineering application problems, reported in many papers； in this report we consider Laplace’ s equation only. In [Li, Huang,

Liaw, Lee*, 2012], entitled as' The null field method of Dirichlet problems of Laplace's equation on circular domains with circular holes' is taken as a basis of this research, some error analysis is given for the assurance of the field nodes to locate on the domain boundary.

As a result, in this project, a new method called the 'interior field method (IFM)' is given, which is the special case of the null field method (NFM) when the field nodes are just located on the domain boundary. The IFM is simpler than the NFM, because only one formula of interior solutions is needed, compared with multiple formulas used in the NFM. The IFM is more advantageous in simplicity and applications. In addition, some flux conservation law allow user to delete one more variable to deduce the number of collocation equation, thus the system of equation can be reduced. Numerical experiments with and without conservative schemes are provided to support the analysis. Some error analysis are also studied in the project.

英文關鍵詞： Boundary integral equations, fundamental solution, Fourier series, Laplace's equations, elliptic PDEs, null field method, interior field method, condition number.

前言：

前年教育部工程科學類獎章得主海洋大學河海工程學系陳正宗終身特聘教授及其研究團隊在近幾年發展了一個應用退化核(degenerate kernel)專門處理橢圓偏微分方程邊界積分方程型態的方法已經成功處理各種應用問題，稱為零場法(Null Field Method-NFM)[ Chen, Shen, Wu, 2005], [Chen et al. 2008c], [Chen, Kuo, Lin, 2002] [Chen, Lee, Chen, Lin, 2002]。零場法在傳統的 Laplace 方程〔Chen, Shen, 2009〕, Helmholtz 方程[Chen et al. 2008a], [Chen et al. 2008b]、biharmonic 方程[Chen, Lee, Liao, 2008]、及 biHelmholtz 方程 [Chen, Liao, Lee, 2009]計算上非常成功。該團隊宣稱這個方法是屬於半解析解法，因為所有的誤差是取有限項的傅立葉級數解所造成，然而由於傳統布點的方式會造成病態矩陣，所以許多使用者偏向直接使用邊界布點的方式，縱使有時邊界有奇異的角點或是不同邊界條件所造成之奇異性會造成困擾。針對邊界沒有奇異性的問題陳教授有處理的方法[ Chen, Shen, Wu, 2005], [Chen et al. 2008c]。雖然零場法原來的定義其場點只能放置在 Domain 內離邊界一點距離之遠處，而且是不包括邊界的，主持人（通訊作者）與中山大學李子才教授在去年(2012 年)發表在 Engineering Analysis with Boundary Elements [Li, Huang, Liaw, Lee*, 2012]針對零場法在邊界上布點的合理性已經證明是可行的施行方式，而且對於病態性的情形作了理論證明，我們發現如其場點放置在邊界上其數值穩定性最好，若是按照原零場法的方式布點，那離邊界越遠則穩定性越差，所以許多研究者的研究報告都是使用直接在邊界布點的方式處理零場法。我們也發現在邊界上或是場域裡布點不會影響太多數值結果的收斂性。 在零場法的使用上由於奇異核的原因，使用退化核可以對內外場以基本解的級數展開，所以計算上必須分別對此內、外兩場產生場點函數，但是在計算時都是直接布點在邊界上了。在 2012 年發表的文章[Li, Huang, Liaw, Lee*, 2012]也證明了在邊界布點的同質性，所以主持人認為可以只要產生一組場點函數，這個場點函數我們暫時稱為內場法，使用時直接將其推展到內外圓的邊界即可滿足問題本身的邊界條件，而不需對內、外場函數同時滿足才能找出所有未知係數。這個研究是將目前盛行的零場法做一個新的解釋，對於使用者來說，主持人仍需要更多資訊，包括使用內場法的收斂性及數值解穩定性等足以與零場法比較的參數才行，也讓工程師在使用邊界積分法時有更多的選擇。在改善計算效率部分，我們也發現了一個原本就滿足的物理性質 Flux 守恆條件，可以將目前的計算，不管是零場法或是內場法都可以產生減少一個變數的條件，使得經過邊界布點後經由 collocation 方程式產生的代數方程式少一個，雖不足以減少太多計算量，但是滿足這部分的物理性質還是很重要的。所以產生一個新的內場法及其守恆型方程及其收斂性分析，及探討對同心圓模型問題其數值方法的穩定性等，這些部分是主持人這個計畫主要內容。

研究過程

**A.原始問題及目前成果說明**

**1.問題說明：Laplace's equations in 2D**

In order to simply describe the Null Field Method (NFM), we confine ourselves Laplace's equation and choose the circular domain with one circular hole. Denote the disks $S_R$ and $S_{R_1}$ with radii $R$ and $R_1$, respectively. Let $S_{R_1} \subset S_R$, and the eccentric circular domains $S_R$ and SR1 may have different origins. Let $2R_1 < R$. The annular solution domain S = $S_R \setminus S_{R_1}$ with the exterior and the interior boundaries $\partial S_R$ and $\partial S_{R_1}$, respectively. In [Chen and Shen, 2009], $R=2.5$, and $R_1=1$, and the origins of $S_R$ and $S_{R_1}$ are located at (0,0) and $(-R_1,0)$ respectively. The following Dirichlet problems are discussed by Palaniappan [Palaniappan, 2002],

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \ \text{in } S, \tag{1}$$

$$u=1, \ \text{on } \partial S_R \tag{2}$$

$$u=0, \ \text{on } \partial S_{R_1} \tag{3}$$

The true solution of Equations 1-2 can be found at [Palaniappan,2002] and is given as,

$$u(\rho,\varphi)=\frac{1}{2\ln 2}\ln\{\frac{16\rho^2+1+8\rho\cos\varphi}{\rho^2+16+8\rho\cos\varphi}\}, \tag{4}$$

where $(\rho,\varphi)$ are the polar coordinate of $S_{R_1}$ with origin $(-1,0)$. Equations (1-3) are called the Model problem in this research.

We assume the boundary conditions are given and can be approximated by Fourier expansions.

On the exterior boundary $\partial S_R$, there exist the approximation of Fourier expansions,

$$u=u_0:=a_0+\sum_{k=1}^{M}\{a_k\cos k\theta+b_k\sin k\theta\}, \quad \text{on } \partial S_R \tag{5}$$

$$\frac{\partial u}{\partial \nu}=q_0:=p_0+\sum_{k=1}^{M}\{p_k\cos k\theta+q_k\sin k\theta\}, \ \text{on } \partial S_R \tag{6}$$

where $a_k,b_k,p_k,q_k$ are coefficients.

On the interior boundary $\partial S_{R_1}$, there exist the approximation of Fourier expansions,

$$\bar{u}=\bar{u}_0:=\bar{a}_0+\sum_{k=1}^{M}\{\bar{a}_k\cos k\bar{\theta}+\bar{b}_k\sin k\bar{\theta}\}, \quad \text{on } \partial S_{R_1} \tag{7}$$

$$\frac{\partial \bar{u}}{\partial \bar{\nu}}=\bar{q}_0:=\bar{p}_0+\sum_{k=1}^{M}\{\bar{p}_k\cos k\bar{\theta}+\bar{q}_k\sin k\bar{\theta}\}, \ \text{on } \partial S_{R_1} \tag{8}$$

where $\bar{a}_k,\bar{b}_k,\bar{p}_k,\bar{q}_k$ are coefficients. In Equations 5-8, $\theta$ and $\bar{\theta}$ are the polar coordinates of $S_R$ and $S_{R_1}$, respectively, and $\nu$ and $\bar{\nu}$ are the outward normal of $\partial S_R$ and $\partial S_{R_1}$, respectively. The Dirichlet conditions, the Neumann conditions, and their mixed types on $\partial S_R$ are given with known coefficients.

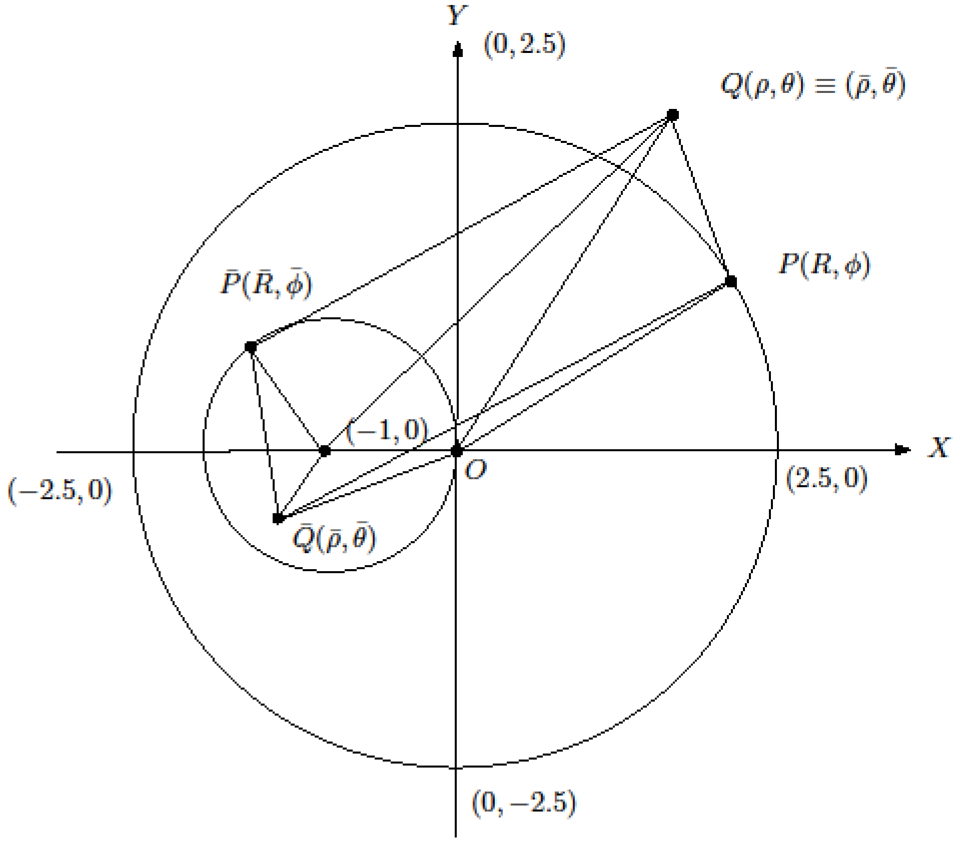


Figure 1: Configuration of the Model problem

In S, we denote two nodes $\boldsymbol{x} = Q = (x, y) = (\rho, \theta)$, and $\boldsymbol{y} = P = (\xi, \eta) = (r, \varphi)$, where $x = \rho\cos\theta$, $y = \rho\sin\theta$; and $\xi = R\cos\varphi$, $\eta = R\sin\varphi$, where $\rho = \sqrt{x^2 + y^2}$ and $R = r$ with $r = \sqrt{\xi^2 + \eta^2}$. The fundamental solution (FS) for Laplace equation is that satisfies the following equation,

$$\nabla^2 \boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y}) = 2\pi\delta(\boldsymbol{x} - \boldsymbol{y})$$

where $\delta(\boldsymbol{x} - \boldsymbol{y})$ denotes the direc-delta function. We obtain the fundamental solution as $\boldsymbol{U}(\boldsymbol{x}, \boldsymbol{y}) = \ln\left|\overline{\boldsymbol{PQ}}\right|$ and $\ln\left|\overline{PQ}\right| = \ln\sqrt{\rho^2 - 2\rho R\cos(\theta - \varphi) + R^2}$, where $\overline{\boldsymbol{PQ}}$ is the line connecting points P and Q [Chen, Shen, Wu, 2005].

From the Boundary Element Method (BEM) Theory [Chen, Zhou, 1992][Yu, 2002], we have three different Green formulas for different location of the field nodes $\boldsymbol{Q}(\boldsymbol{x})$:

$$\int_{\partial S} \{\ln\left|\overline{PQ}\right| \frac{\partial u(y)}{\partial \nu} - u(y)\frac{\partial}{\partial \nu}\ln\left|\overline{PQ}\right|\} d\sigma_y = \begin{cases} -2\pi u(Q), & Q \in S \\ -\pi u(Q), & Q \in \partial S \\ 0, & otherwise \end{cases} \qquad (9)$$

where $\boldsymbol{P}(\boldsymbol{y}) \in (S \cup \partial S)$.

By [Abramowitz et al. 1964], [Li, 2009], [Li, Huang, Huang, 2010], two kinds of series expansions of the FS $\ln\left|\overline{\boldsymbol{PQ}}\right|$ are given as:

$$\ln\left|\overline{\boldsymbol{PQ}}\right| = \ln\left|\boldsymbol{P}(\boldsymbol{y}) - \boldsymbol{Q}(\boldsymbol{x})\right| = \ln\left|Q(\rho,\theta) - P(r,\varphi)\right|$$

$$= \begin{cases} U^{i}(x,y) = \ln r - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{r}\right)^{n} \cos n(\theta-\varphi), & \rho < r \\ U^{e}(x,y) = \ln \rho - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{\rho}\right)^{n} \cos n(\theta-\varphi), & \rho > r \end{cases} \qquad (10)$$

where $\boldsymbol{x} = Q = (x,y) = (\rho,\theta)$ and $\boldsymbol{y} = P = (\xi,\eta) = (r,\varphi)$; and the superscripts "e", and "i" mean the exterior and interior field nodes $\boldsymbol{x}$, respectively.

By Equation 10, we can derive two kinds of derivatives expansions of FS as the following:

$$\begin{cases} \frac{\partial}{\partial r} U^{i}(x,y) = \frac{1}{r} + \sum_{n=1}^{\infty} \left(\frac{\rho^{n}}{r^{n+1}}\right) \cos n(\theta-\varphi), & \rho < r \\ \frac{\partial}{\partial r} U^{e}(x,y) = -\sum_{n=1}^{\infty} \left(\frac{r^{n-1}}{\rho^{n}}\right) \cos n(\theta-\varphi), & \rho > r \end{cases} \qquad (11)$$

Note that the traditional boundary element method is based on the first formula of Equation 9. By moving the field node to the boundary, some Cauchy principle value, Riemann principle value on Laplace equations, and some Hadamard principle value may happen in biHarmonic equations [Chen, Lee, Lee, 2009]. The calculation becomes very difficult, thus the dual null field integral equation is used and the field nodes $\boldsymbol{x}$ are located on the complimentary domain of $S \cup \partial S$. That is called the Null Field Method which is based on the third formula of Equation 9.

So, now we can define the null field method for Laplace's equation that we discuss in our proposal as the following:

$$\int_{\partial S_R \cup \partial S_{R_1}} U(x,y) \frac{\partial u(y)}{\partial \nu} d\sigma_y = \int_{\partial S_R \cup \partial S_{R_1}} u(y) \frac{\partial}{\partial \nu} U(x,y) d\sigma_y, \qquad \boldsymbol{x} \in \bar{S}^{C} \qquad (12)$$

where $\bar{S}^{C}$ is the compliment of domain $S \cup \partial S$.

By substituting the Fourier expansions of the Fundamental solutions in Equation 10 and its derivative in Equation 11 into Equation 12 yield the basic algorithm of the Null Field Method, noting that the normal derivative of interior circle $\partial S_{R_1}$ is defined by $\frac{\partial U(x,y)}{\partial \nu} = -\frac{\partial U(x,y)}{\partial r}$. The reason of naming the Null Field is because that the field nods $\boldsymbol{x}$ or Q is supposed to locate outside of the solution domain S and its boundary $\partial S$ only.

Chen and his research group have applied the Null Field Method to solve many engineering applications, and the governing equations include Laplace' s equation, Poisson Equations, BiHarmonic equation, Helmholtz equation, etc. ([Chen, Shen, Wu 2005], [Chen et al. 2008a, b, c], [Chen, Lee, Liao, 2008]、[Chen, Kuo, Lin, 2002], [Chen, Lee, Chen Lin, 2002], [Chen, Lee, Liao, 2008],[Chen, Shen 2009], [Chen, Liao, Lee,2009],[Chen, Lee, Lee, 2009], etc). Although the basic descriptions above have been used in many papers of Chen, there exist no complete explicit equations reported so far. The explicit equations are important not only to understand the intrinsic nature of the NFM, but also to extend their applications. In the Green formula Equation 9, the field node Q $= \boldsymbol{x} = (\rho, \theta)$ is supposed to locate outside of the solution domain $S \cup \partial S$ only; this is why the algorithms of Chen is called the null field method (NFM). Some new evidences have been observed in the new article **『The null-field method of Dirichlet problems of Laplace' s equation on circular domains with circular holes』** by Li, Huang, Liao, and Lee, and has been published at Engineering Analysis with Boundary Elements, 36(2012), pp477-491, including that we can locate the field node just on the domain boundary: x∈∂S; Also a better choice of location of x is also suggested by numerical experiment.

In order to describe two systems of polar coordinates by $(\rho, \theta)$ and $(\bar{\rho}, \bar{\theta})$ with origins (0,0) and $(x_1, y_1)$ for $S_R$ and $S_{R_1}$, respectively ( see Figure 1). The conversion between these two polar coordinates is defined as:

$$\rho = \sqrt{(\bar{\rho}\cos(\bar{\theta}) + x_1)^2 + (\bar{\rho}\sin(\bar{\theta}) + y_1)^2} \quad \text{and} \quad \tan\theta = \frac{\bar{\rho}\sin(\bar{\theta}) + y_1}{\bar{\rho}\cos(\bar{\theta}) + x_1} \tag{13}$$

$$\bar{\rho} = \sqrt{(\rho\cos(\theta) - x_1)^2 + (\rho\sin(\theta) - y_1)^2} \quad \text{and} \quad \tan\bar{\theta} = \frac{\rho\sin(\theta) - y_1}{\rho\cos(\theta) - x_1} \tag{14}$$

By referring to [Li, Huang, Liaw, Lee*, 2012], we have established the first explicit algebraic equations of NFM as:

For the exterior field node $\boldsymbol{x} = (\rho, \theta)$ with $\rho > r = R$, we have

$$\begin{aligned} L_{ext}(\rho,\theta,\bar{\rho},\bar{\theta}) :=& -R\pi\sum_{k=1}^{M}\left(\frac{R^{k-1}}{\rho^k}\right)(a_k\cos(k\theta)+b_k\sin(k\theta))+ \\ & R_1\pi\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\bar{\rho}^k}\right)(\bar{a}_k\cos(k\bar{\theta})+\bar{b}_k\sin(k\bar{\theta}))-\{2\pi R(\ln\rho)p_0- \\ & R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{R}{\rho}\right)^k(p_k\cos(k\theta)+q_k\sin(k\theta))+2\pi R_1(\ln\bar{\rho})\bar{p}_0- \\ & R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\bar{\rho}}\right)^k(\bar{p}_k\cos(k\bar{\theta})+\bar{q}_k\sin(k\bar{\theta}))\}=0 \end{aligned} \tag{15}$$

For the interior field node $\boldsymbol{x} = (\bar{\rho}, \bar{\theta})$ with $\bar{\rho} < \bar{r} = R_1$, we have

$$\begin{aligned} L_{\text{int}}(\rho,\theta,\bar{\rho},\bar{\theta}) :=& -2\pi\bar{a}_0 - R_1\pi\sum_{k=1}^{N}\left(\frac{\bar{\rho}^k}{R_1^{k+1}}\right)(\bar{a}_k\cos(k\bar{\theta})+\bar{b}_k\sin(k\bar{\theta}))+2\pi a_0 \\ & R\pi\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)(a_k\cos(k\theta)+b_k\sin(k\theta))-\{2\pi R_1(\ln R_1)\bar{p}_0- \\ & R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{\bar{\rho}}{R_1}\right)^k(\bar{p}_k\cos(k\bar{\theta})+\bar{q}_k\sin(k\bar{\theta}))+2\pi R(\ln R)p_0- \\ & R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k(p_k\cos(k\theta)+q_k\sin(k\theta))\}=0 \end{aligned} \tag{16}$$

Equations 15 and 16 are called the explicit algebraic equations of NFM.

**模型問題的零場法**

## 2. Null Field Method for the Model Problem

### (2.1) The collocation equations for the conventional null field method

Since one of the Dirichlet or the Neumann conditions is given on $\partial S_R$ and $\partial S_{R_1}$, only 2(M+N)+2 coefficients in Equations 15-16 are unknown. We choose 2M+1 and 2N+1 field nodes on $\bar{S}^C$, which may be located uniformly on the exterior and the interior circles, as shown in Figure 2.

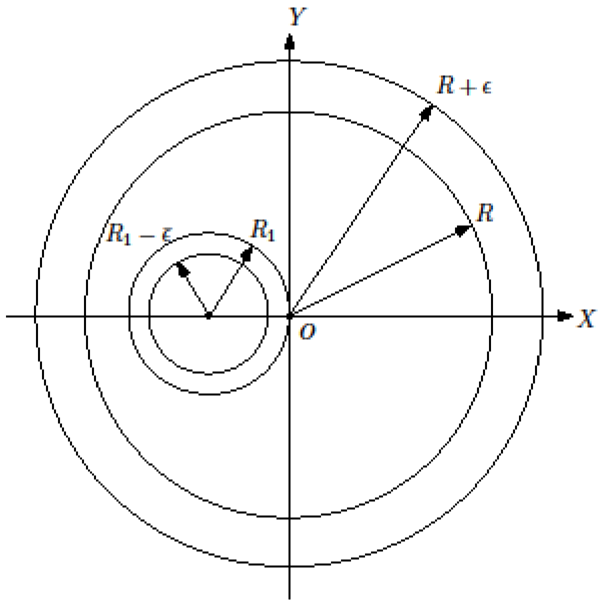


Figure 2. Location of field nodes on the exterior of large circle and interior of the small circle

$(\rho,\theta)=(R+\varepsilon,i\Delta\theta)$, i=0, 1, 2, ⋯, 2M

$(\overline{\rho},\overline{\theta})=(R_1-\overline{\varepsilon},i\Delta\overline{\theta})$, i=0, 1, 2, ⋯, 2N

where $\varepsilon \geq 0, 0 \leq \overline{\varepsilon} < R_1$, and $\Delta\theta=\frac{2\pi}{2M+1}, \Delta\overline{\theta}=\frac{2\pi}{2N+1}$.

A system of 2(M+N)+2 discrete equations from Equations 15-16 can be obtained as:

$$L_{ext}(R+\varepsilon,i\Delta\theta,\overline{\rho}_i,\overline{\theta}_i)=0,\ \ i=0,1,2,...,2M \tag{17}$$

$$L_{\text{int}}(\rho_i,\theta,R_1-\overline{\varepsilon},i\Delta\overline{\theta})=0,\ \ i=0,1,2,...,2N \tag{18}$$

The corresponding $(\rho_i,\theta_i)$ and $(\overline{\rho}_i,\overline{\theta}_i)$ can be obtained from the conversion formula of Equations 13-14, respectively. Thus a linear system of algebraic equations is obtained and can be written as

$$\boldsymbol{Fz}=\boldsymbol{b}, \tag{19}$$

where the matrices $\boldsymbol{F}\in \boldsymbol{R}^{n\times n}$, and n=2(M+N)+2. The unknown coefficients can be obtained by Equation 19. Once all the coefficients are available, the solution at the interior domain, which is based on the first equation of the Green formula of Equation 9, the solution at the interior node $(\rho,\theta)\in S$ can be written as:

$$\boldsymbol{u}(\boldsymbol{x}) = u(\rho,\theta) = \frac{-1}{2\pi}\int_{\partial S_R \cup \partial S_{R_1}} \left\{ U(x,y)\frac{\partial u(y)}{\partial \nu} - u(y)\frac{\partial U(x,y)}{\partial r} \right\} d\sigma_y \tag{20}$$

By some manipulation, the solution can be written as:

$$\begin{aligned} u_{M-N} &= u_{M-N}(\rho,\theta) = u_{M-N}(\bar{\rho},\bar{\theta}) = a_0 - R(\ln R)p_0 - R_1(\ln R_1)\bar{p}_0 \\ &+ \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho^k}{R^k}\right)(p_k\cos(k\theta) + q_k\sin(k\theta)) + \frac{R}{2}\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)(a_k\cos(k\theta) + b_k\sin(k\theta)) \quad , (\rho,\theta)\in S \\ &+ \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\bar{\rho}}\right)^k(\bar{p}_k\cos(k\bar{\theta}) + \bar{q}_k\sin(k\bar{\theta})) + \frac{R_1}{2}\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\bar{\rho}^k}\right)(\bar{a}_k\cos(k\bar{\theta}) + \bar{b}_k\sin(k\bar{\theta})) \end{aligned} \tag{21}$$

and $(\bar{\rho},\bar{\theta})\in S$ are available from conversion of Equation 9.

### (2.2) Main theoretical results

The main theoretical result in [Li, Huang, Liaw, Lee*, 2012] is given to re-assure the location of field nodes can be at the boundary.

Theorem A. Let $u \in H^p(\partial S) \cup u_\nu \in H^{p-1}(\partial S) (p \geq 2)$ be given, then Equations 15-16 hold for $\rho \to R$ and $\bar{\rho} \to R_1$, respectively.

So, the field nodes can now be located on the boundary on the implementation of NFM without worrying about the use of the Green formula given at Equation 9. Since in many engineering applications of NFM, users usually put the field nodes on the boundary, and they may not know the reason exactly, so we actually solve one of their puzzle.

### (2.3) Stability criterion

What if we still put the field nodes on the complimentary domain $\bar{S}^C$, that is when $\varepsilon > 0,\ 0 < \bar{\varepsilon} < R_1$? What actually the numerical results will show? The second theoretical result of [Li, Huang, Liaw, Lee*, 2012] gave the conclusion that the choices of $(\varepsilon,\bar{\varepsilon})$ in the NFM do not affect much on convergence rates, but do have influence on stability. Let $\delta$ denote the distance of field node Q to $\partial S$. The larger $\delta$ is chosen, the worse the instability of the NFM occurs. As a result, $\delta$=0 (i.e., Q $\in \partial S$) is the best for stability. However, when $\delta$>0, the errors are slightly smaller. Therefore, small $\delta$ is a favorable choice for both high accuracy and good stability.

From above analysis, we know basically some theoretical results of implementing the conventional Null Field Method and since it is so widely used in engineering community, the above results are important for users.

A conservative scheme of the above mentioned methods is further studied, which is called the conservative scheme. Some stability analysis of the collocation schemes of the above mentioned methods and error analysis have been studied. Further study of the null field method, such as the Interior Field Method can provide users more choices on implementation when they solve boundary value problems.

In this project, we have gievn some important results of the conservative scheme of the null field method and the introduction of a new method called Interior Field Method, which is a special case of Null Field Method.

## B、計畫研究方法、進行步驟

**A Conservative Scheme of Null Field Method and Interior Field Method**

**for Laplace’s Equation on Dirichlet Boundary Condition**

### (1) A conservative scheme

In some physical problems, the flux conservation is imperative and essential. The conservative scheme of NFM can be designed to satisfy exactly the flux conservation,

$$\int_{S_R} (u_M)_\nu + \int_{S_{R_1}} (u_N)_{\bar{\nu}} = 0 \qquad (22)$$

where $u_M$ is given at Equation 6, and is $\frac{\partial u}{\partial \nu} = q_0 := p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\}$; and $u_N$ is given at Equation 8, and is $\frac{\partial \bar{u}}{\partial \bar{\nu}} = \bar{q}_0 := \bar{p}_0 + \sum_{k=1}^{M} \{\bar{p}_k \cos k\bar{\theta} + \bar{q}_k \sin k\bar{\theta}\}$. So just substitute these two function into Equation 22, we have an the result directly,

$$2\pi R p_0 + 2\pi R_1 \overline{p}_0 = 0 \tag{23}$$

or

$$R p_0 + R_1 \overline{p}_0 = 0 \tag{24}$$

By Equation 23 or Equation 24, one variable can be substituted by another, say that,

$$\overline{p}_0 = -\frac{R}{R_1} p_0 \tag{25}$$

Therefore, the system of NFM in Equations 15-16 can be rewritten as the following,

$$\begin{aligned} L_{ext}^{Conserv}(\rho,\theta,\overline{\rho},\overline{\theta}) := & -R\pi\sum_{k=1}^{M}\left(\frac{R^{k-1}}{\rho^k}\right)(a_k\cos(k\theta)+b_k\sin(k\theta))+ \\ & R_1\pi\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\overline{\rho}^k}\right)(\overline{a}_k\cos(k\overline{\theta})+\overline{b}_k\sin(k\overline{\theta}))-\{2\pi R(\ln\frac{\rho}{\overline{\rho}})p_0- \\ & R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{R}{\rho}\right)^k(p_k\cos(k\theta)+q_k\sin(k\theta))- \\ & R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k(\overline{p}_k\cos(k\overline{\theta})+\overline{q}_k\sin(k\overline{\theta}))\}=0 \end{aligned} \tag{26-1}$$

$$\begin{aligned} L_{\text{int}}^{Conserv}(\rho,\theta,\overline{\rho},\overline{\theta}) := & -2\pi\overline{a}_0 - R_1\pi\sum_{k=1}^{N}\left(\frac{\overline{\rho}^k}{R_1^{k+1}}\right)(\overline{a}_k\cos(k\overline{\theta})+\overline{b}_k\sin(k\overline{\theta}))+2\pi a_0+ \\ & R\pi\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)(a_k\cos(k\theta)+b_k\sin(k\theta))-\{2\pi R(\ln\frac{R}{R_1})p_0- \\ & R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k(p_k\cos(k\theta)+q_k\sin(k\theta))- \\ & R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{\overline{\rho}}{R_1}\right)^k(\overline{p}_k\cos(k\overline{\theta})+\overline{q}_k\sin(k\overline{\theta}))\}=0 \end{aligned} \tag{26-2}$$

The new collocation scheme can thus be designed is similar to Equations 17-18, and can be rewritten as:

**Conservative Scheme :**

A new system of 2(M+N)+1 equations, with 2(M+N)+1 field nodes,

$$w_i L_{ext}^{Conserv}(R+\varepsilon, i\Delta\theta, \bar{\rho}_i, \bar{\theta}_i) = 0, \quad i = 0,1,2,\ldots,2M \qquad (27)$$

$$w_i L_{\text{int}}^{Conserv}(\rho_i, \theta, R_1 - \bar{\varepsilon}, i\Delta\bar{\theta}) = 0, \quad i = 0,1,2,\ldots,2N-1 \qquad (28)$$

where $w_i$ are some weight function to balance the different boundary conditions.

It means that when we just use the same number of field nodes with the number of unknown variables The first question we need to know is the performance of the numerical results, do they have good convergence rate and good stability as in the traditional NFM?
From the output of the numerical results in Table 1, For Model Problem, choose the better match between M and N as (M,N) = (2:1). The conservative
scheme with $\epsilon = \bar \epsilon =0$ are used in computation ( whih is the common way of implementation of the NFM). The domain errors and the errors of normal derivatives on the boundary are defined in [Li, Huang, Liaw, Lee*, 2012]. The errors and condition numbers for different (M,N) with $(\epsilon, \bar \epsilon) = (0,0)$ are listed in Table 1 and their leading coefficients $\p_k$ and $\bar\p_k$ are given in Tables 2 and 3, respectively.

Table 1: Errors and condition numbers for Model Problem by the conservative schemes with $M = 2N$ and $\epsilon = \bar{\epsilon} = 0$.

| $(M, N)$ | (4, 2) | (10, 5) | (20, 10) | (30, 15) | (40, 20) | (50, 25) | (60, 30) |
|---|---|---|---|---|---|---|---|
| $\|\|u - u_{M-N}\|\|_{\infty,S}$ | 1.05(-1) | 1.29(-3) | 1.06(-6) | 9.95(-10) | 9.83(-13) | 1.00(-15) | 1.06(-18) |
| $\|\|u - u_{M-N}\|\|_{0,\partial S}$ | 1.74(-1) | 2.07(-3) | 1.74(-6) | 1.68(-9) | 1.69(-12) | 1.75(-15) | 1.85(-18) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{\infty,\partial S}$ | 5.43(-1) | 1.33(-2) | 2.12(-5) | 2.98(-8) | 3.93(-11) | 5.01(-14) | 6.38(-17) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{0,\partial S}$ | 8.03(-1) | 2.09(-2) | 3.49(-5) | 5.02(-8) | 6.77(-11) | 8.77(-14) | 1.11(-16) |
| $\sigma_{\max}$ | 5.53 | 8.66 | 12.2 | 1.49(1) | 1.71(1) | 1.92(1) | 2.11(1) |
| $\sigma_{\min}$ | 1.58(-3) | 9.77(-6) | 4.23(-9) | 2.40(-12) | 1.52(-15) | 1.03(-18) | 7.25(-22) |
| $\sigma_{min-next}$ | 4.35(-1) | 2.75(-1) | 1.76(-1) | 1.38(-1) | 1.18(-1) | 1.04(-1) | 9.44(-2) |
| Cond | 3.50(3) | 8.86(5) | 2.88(9) | 6.22(12) | 1.13(16) | 1.87(19) | 2.91(22) |
| Cond_eff | 1.13(3) | 1.75(5) | 5.72(8) | 1.23(12) | 2.25(15) | 3.71(18) | 5.78(21) |

Table 2: The coefficients $p_k$ for Model Problem by the conservative schemes with $(M, N) = (60, 30)$ and $\epsilon = \bar{\epsilon} = 0$.

| $k$ | $p_k$ | $k$ | $p_k$ |
|---|---|---|---|
| 0 | 0.5770780163555853625933433 | 31 | -5.374457839373229664929012(-10) |
| 1 | -0.5770780163555853622496607 | 32 | 2.687228919657442755329359(-10) |
| 2 | 0.2885390081777926807976638 | 33 | -1.343614459804690596367250(-10) |
| 3 | -0.1442695040888963400926696 | 34 | 6.718072298825661844946376(-11) |
| 4 | 0.07213475204444816976408510 | 35 | -3.359036149250162612080399(-11) |
| 5 | -0.03606737602222408462520972 | 36 | 1.679518074491399723174677(-11) |
| 6 | 0.01803368801111204208153934 | 37 | -8.397590371359200011826765(-12) |
| 7 | -0.009016844005556020834934555 | 38 | 4.198795184778737912560074(-12) |
| 8 | 0.004508422002778010235687519 | 39 | -2.099397591650664663876194(-12) |
| 9 | -0.002254211001389004958521401 | 40 | 1.049698795220068984390129(-12) |
| 10 | 0.001127105500694502340550229 | 41 | -5.248493971144989136138502(-13) |
| 11 | -0.0005635527503472510502195915 | 42 | 2.624246981518089173529479(-13) |
| 12 | 0.0002817763751736254217417780 | 43 | -1.312123487441613418977099(-13) |
| 13 | -0.0001408881875868126222836058 | 44 | 6.560617410002066841989570(-14) |
| 14 | 0.00007044409379340623553631763 | 45 | -3.280308682499658944945444(-14) |
| 15 | -0.00003522204689670305348186035 | 46 | 1.640154322182716998703010(-14) |
| 16 | 0.00001761102344835147226200443 | 47 | -8.200771439360115784056128(-15) |
| 17 | -8.805511724175690102745908(-6) | 48 | 4.100385544863161885186935(-15) |
| 18 | 4.402755862087806269441078(-6) | 49 | -2.050192555590879377263215(-15) |
| 19 | -2.201377931043870539709053(-6) | 50 | 1.025095945011031712555171(-15) |
| 20 | 1.100688965521907937032914(-6) | 51 | -5.125473760988550788688903(-16) |
| 21 | -5.503444827609310960441710(-7) | 52 | 2.562725269619909724745545(-16) |
| 22 | 2.751722413804464446009859(-7) | 53 | -1.281339188110428401643500(-16) |
| 23 | -1.375861206902072949192052(-7) | 54 | 6.406215453481607042947389(-17) |
| 24 | 6.879306034509038966114312(-8) | 55 | -3.202117537983115593265803(-17) |
| 25 | -3.439653017253417576801460(-8) | 56 | 1.599013974395542642025499(-17) |
| 26 | 1.719826508625794230818684(-8) | 57 | -7.952813198425345073524545(-18) |
| 27 | -8.599132543121390392021213(-9) | 58 | 3.889060742356455201052180(-18) |
| 28 | 4.299566271554419120316275(-9) | 59 | -1.763972137438824022366321(-18) |
| 29 | -2.149783135772019511589494(-9) | 60 | 5.087371094498464587786628(-19) |
| 30 | 1.074891567881722357283158(-9) | | |

Table 3: The coefficients $\bar{p}_k$ for Model Problem by the conservative schemes with $(M, N) = (60, 30)$ and $\epsilon = \bar{\epsilon} = 0$.

| $k$ | $\bar{p}_k$ | $k$ | $\bar{p}_k$ |
|---|---|---|---|
| 1 | 0.72134752044448170109366 30 | 16 | -6.71807206601309471267698 4(-10) |
| 2 | -0.18033688011112042230945 41 | 17 | 1.67951782693509611336200 2(-10) |
| 3 | 0.04508422002778010169383850 | 18 | -4.19879256229110166690683 1(-11) |
| 4 | -0.01127105500694502055048827 | 19 | 1.04969602616065005294670 3(-11) |
| 5 | 0.00281776375173624923119316 3 | 20 | -2.62421782762252967777150 2(-12) |
| 6 | -0.00070444093793405534153163 99 | 21 | 6.56031125467825586611178 7(-13) |
| 7 | 0.00017611023448350579404088 07 | 22 | -1.63983356272641397326895 0(-13) |
| 8 | -0.00004402755862086732353848251 | 23 | 4.09703203297453400797119 4(-14) |
| 9 | 0.00001100688965520661758726790 | 24 | -1.02159677545588160104133 4(-14) |
| 10 | -2.75172241379035024192954 3(-6) | 25 | 2.52628625589052631331719 8(-15) |
| 11 | 6.87930603435191202733959 0(-7) | 26 | -6.02773530000630134120241 0(-16) |
| 12 | -1.71982650845308533817144 2(-7) | 27 | 1.20807287323401089704820 7(-16) |
| 13 | 4.29956626967445904252389 1(-8) | 28 | 7.54005427179807175759885 5(-19) |
| 14 | -1.07489156585101392766967 7(-8) | 29 | -3.21357700668914671175785 0(-17) |
| 15 | 2.68722889785796443124350 2(-9) | 30 | 4.06382826630277239248373 4(-17) |

From Table 1, we can see the following asymptotes

$$\|u - (u_{M-N})\|_{\infty,S} = O(0.504^M), \quad \|u_\nu - (\hat{u}_{M-N})_\nu\|_{\infty,\partial S} = O(0.503^M),$$
$$\text{Cond} = O(1.94^M).$$

Errors in Table 1 are almost the same as those in [Li, Huang, Liaw, Lee*, 2012], that means the convergence rate are almost the same. But the condition number in Table 1 is much larger than $\mbox{Cond} = O(M)$ as shown in [Li, Huang, Liaw, Lee*, 2012]. Compared Tables 2 and Table 3 with those in [Li, Huang, Liaw, Lee*, 2012], we can see that the leading coefficients $p_0$ and $p_1$ also have the same 18 significant digits. Since $\bar p_0 =-2.5 p_0$, the leading coefficient $\bar p_0$ also has the same 18 significant digits exactly. From the computed results, we conclude that the accuracy of the solutions by the conservative schemes retains the optimal convergence rates, but the stability is unexpectedly deteriorating as $M$ increases! Let us scrutinize the numerical singular values $\sigma_i$, there exist the inequalities,

$$0 < \sigma_{\min}(= \sigma_n) \ll \sigma_i, \quad i = 1, 2, ..., n-1.$$

Only the infinitesimal $\sigma_{\min}$ causes the severe instability of discrete matrices, which is called the pseudo-singularity in this report. **This is a new kind of numerical instability found in the collocation Trefftz method.** Next, in order to overcome this kind of instability, we use the following over-determined system with collocation equations on the circles of $\partial S_R$ and $\partial S_{R_1}$, which were used for the original NFM equations in [Li, Huang, Liaw, Lee*, 2012],

**Overdetermined Scheme:**

A new system of 2(M+N)+1 equations, with 2(M+N)+2 field nodes,

$$w_i L_{ext}^{Conserv}(R+\varepsilon, i\Delta\theta, \bar{\rho}_i, \bar{\theta}_i) = 0, \quad i = 0,1,2,\ldots,2M \tag{29}$$

$$w_i L_{\text{int}}^{Conserv}(\rho_i, \theta, R_1 - \bar{\varepsilon}, i\Delta\bar{\theta}) = 0, \quad i = 0,1,2,\ldots,2N \tag{30}$$

where $w_i$ are some weight function to balance the different boundary conditions.

For $(\epsilon, \bar \epsilon)= (0,0)$, the numerical results are given in Table 4, we can see

$$\|u - (u_{M-N})\|_{\infty,S} = O(0.494^M), \quad \|u_\nu - (\hat{u}_{M-N})_\nu\|_{\infty,\partial S} = O(0.491^M),$$
$$\text{Cond} = O(M), \quad \text{Cond\_eff} = O(M).$$

Remarkably, the good stability of Cond $= O(M)$ is recovered successfully, while the high convergence rates retain. Compared Tables 4 with those in [Li, Huang, Liaw, Lee*, 2012], both errors and condition numbers are small. Note that all coefficients $p_k$ and $\bar p_k$ in Tables 5 and 6 monotonously decrease in magnitude, but the last few coefficients $\bar p_k (k = 29, 30)$ in Table 3 alter slightly increasingly in magnitude due to the severe instability (i.e., the pseudo-singularity). Hence, the instability is successfully recovred by using the over-determined system Equations 29 and 30.

Table 4: Errors and condition numbers for Model Problem by the conservative schemes with $M = 2N$ and $\epsilon = \bar{\epsilon} = 0$ by the over-determined system.

| $(M, N)$ | (4, 2) | (10, 5) | (20, 10) | (30, 15) | (40, 20) | (50, 25) | (60, 30) |
|---|---|---|---|---|---|---|---|
| $\|\|u - u_{M-N}\|\|_{\infty,S}$ | 2.65(-2) | 2.29(-4) | 1.24(-7) | 8.24(-11) | 5.95(-14) | 4.58(-17) | 3.95(-20) |
| $\|\|u - u_{M-N}\|\|_{0,\partial S}$ | 3.68(-2) | 3.11(-4) | 1.68(-7) | 1.12(-10) | 8.31(-14) | 6.54(-17) | 5.35(-20) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{\infty,\partial S}$ | 1.26(-1) | 2.05(-3) | 2.04(-6) | 2.01(-9) | 1.93(-12) | 1.85(-15) | 1.91(-18) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{0,\partial S}$ | 1.85(-1) | 2.96(-3) | 2.96(-6) | 2.93(-9) | 2.88(-12) | 2.82(-15) | 2.76(-18) |
| $\sigma_{\max}$ | 5.99 | 8.96 | 1.24(1) | 1.51(1) | 1.74(1) | 1.94(1) | 2.12(1) |
| $\sigma_{\min}$ | 3.54(-1) | 2.34(-1) | 1.63(-1) | 1.31(-1) | 1.13(-1) | 1.01(-1) | 9.20(-2) |
| Cond | 1.69(1) | 3.83(1) | 7.66(1) | 1.15(2) | 1.53(2) | 1.92(2) | 2.30(2) |
| Cond_eff | 5.57 | 1.23(1) | 2.45(1) | 3.68(1) | 4.90(1) | 6.13(1) | 7.35(1) |

Table 5: The coefficients $p_k$ for Model Problem by the conservative schemes with $(M, N) = (60, 30)$ and $\epsilon = \bar{\epsilon} = 0$ by the over-determined system.

| $k$ | $p_k$ | $k$ | $p_k$ |
|---|---|---|---|
| 0 | 0.5770780163555853629439699 | 31 | -5.374457839462769797566900(-10) |
| 1 | -0.5770780163555853629439699 | 32 | 2.687228919731384888542087(-10) |
| 2 | 0.2885390081777926814719849 | 33 | -1.343614459865692423072643(-10) |
| 3 | -0.1442695040888963407359925 | 34 | 6.718072299328461683518886(-11) |
| 4 | 0.07213475204444817036799623 | 35 | -3.359036149664229979853775(-11) |
| 5 | -0.03606737602222408518399812 | 36 | 1.679518074832113305708264(-11) |
| 6 | 0.01803368801111204259199906 | 37 | -8.397590374160534189348973(-12) |
| 7 | -0.009016844005556021295999529 | 38 | 4.198795187080205638250550(-12) |
| 8 | 0.004508422002778010647999765 | 39 | -2.099397593539986192581286(-12) |
| 9 | -0.002254211001389005323999882 | 40 | 1.049698796769770195339749(-12) |
| 10 | 0.001127105500694502661999941 | 41 | -5.248493983844534393743016(-13) |
| 11 | -0.0005635527503472513309999706 | 42 | 2.624246991913775442262780(-13) |
| 12 | 0.0002817763751736256654999853 | 43 | -1.312123495939928592490997(-13) |
| 13 | -0.0001408881875868128327499926 | 44 | 6.560617479356716355426658(-14) |
| 14 | 0.00007044409379340641637499632 | 45 | -3.280308738978606842769966(-14) |
| 15 | -0.00003522204689670320818749816 | 46 | 1.640154368052798450419112(-14) |
| 16 | 0.00001761102344835160409374908 | 47 | -8.200771810665351894154779(-15) |
| 17 | -8.805511724175802046874540(-6) | 48 | 4.100385844218860428969167(-15) |
| 18 | 4.402755862087901023437270(-6) | 49 | -2.050192795789315252377309(-15) |
| 19 | -2.201377931043950511718635(-6) | 50 | 1.025096136668673809894975(-15) |
| 20 | 1.100688965521975255859317(-6) | 51 | -5.125475280374284479266831(-16) |
| 21 | -5.503444827609876279296583(-7) | 52 | 2.562726465097632590296277(-16) |
| 22 | 2.751722413804938139648285(-7) | 53 | -1.281340120397674119089631(-16) |
| 23 | -1.375861206902469069824129(-7) | 54 | 6.406222644561658881927048(-17) |
| 24 | 6.879306034512345349120369(-8) | 55 | -3.202123004881339257912907(-17) |
| 25 | -3.439653017256172674559608(-8) | 56 | 1.599018043568673021094417(-17) |
| 26 | 1.719826508628086337278600(-8) | 57 | -7.952842452856667003836176(-18) |
| 27 | -8.599132543140431686367759(-9) | 58 | 3.889080438465867808393052(-18) |
| 28 | 4.299566271570215843130914(-9) | 59 | -1.763983534969611314000936(-18) |
| 29 | -2.149783135785107921454235(-9) | 60 | 5.087408686431604405217349(-19) |
| 30 | 1.074891567892553960493538(-9) | | |

Table 6: The coefficients $\bar{p}_k$ for Model Problem by the conservative schemes with $(M, N) = (60, 30)$ and $\epsilon = \bar{\epsilon} = 0$ by the over-determined system.

| $k$ | $\bar{p}_k$ | $k$ | $\bar{p}_k$ |
|---|---|---|---|
| 1 | 0.7213475204444817036799623 | 16 | -6.718072299328462242649207(-10) |
| 2 | -0.1803368801111204259199906 | 17 | 1.679518074832115518124494(-10) |
| 3 | 0.04508422002778010647999765 | 18 | -4.198795187080287137406514(-11) |
| 4 | -0.01127105500694502661999941 | 19 | 1.049698796770065049200292(-11) |
| 5 | 0.002817763751736256654999853 | 20 | -2.624246991924881368700807(-12) |
| 6 | -0.0007044409379340641637499632 | 21 | 6.560617479800278347229738(-13) |
| 7 | 0.0001761102344835160409374908 | 22 | -1.640154369899151407001643(-13) |
| 8 | -0.00004402755862087901023437270 | 23 | 4.100385922568245571104681(-14) |
| 9 | 0.00001100688965521975255859318 | 24 | -1.025096471307767525854892(-14) |
| 10 | -2.751722413804938139648293(-6) | 25 | 2.562740778681797250478746(-15) |
| 11 | 6.879306034512345349120710(-7) | 26 | -6.406834851346877639025390(-16) |
| 12 | -1.719826508628086337280097(-7) | 27 | 1.601635617594257072769615(-16) |
| 13 | 4.299566271570215843197466(-8) | 28 | -4.000965180428373746972636(-17) |
| 14 | -1.074891567892553960789913(-8) | 29 | 9.868954592559973501948267(-18) |
| 15 | 2.687228919731384901650330(-9) | 30 | -1.897190579317880522087191(-18) |

In order to test the way it should be to put the field nodes at the exterior of domain, we implement the Model problem by using a match of $(\epsilon, \bar \epsilon) = (0.5, 0.2)$, according to the design given in [Li, Huang, Liaw, Lee*, 2012]. From Table 7, we can find the following asymptotes,

$$\|u - (u_{M-N})\|_{\infty,S} = O(0.493^M), \quad \|u_\nu - (\hat{u}_{M-N})_\nu\|_{\infty,\partial S} = O(0.491^M),$$
$$\text{Cond} = O(M \times 1.2^M).$$

The growth rate of the condition number in Table 7 is the same as that in [Li, Huang, Liaw, Lee*, 2012]. From the computed results, we conclude that good stability of conservative schemes has been restored by using the strategy of the over-determined system, as given in Equations 29 and 30. In fact, we may add more collocation equations (i.e., m >2(M+N)+1 )as done in the collocation Trefftz method (CTM) in [LLHC2008]. The computed results are similar.

Table 7: Errors and condition numbers for Model Problem by the conservative schemes with $M = 2N$ and $(\epsilon, \bar{\epsilon}) = (0.5, 0.2)$ by the over-determined system.

| $(M, N)$ | (4, 2) | (10, 5) | (20, 10) | (30, 15) | (40, 20) | (50, 25) | (60, 30) |
|---|---|---|---|---|---|---|---|
| $\|\|u - u_{M-N}\|\|_{\infty,S}$ | 2.28(-2) | 1.91(-4) | 1.02(-7) | 6.82(-11) | 4.96(-14) | 3.82(-17) | 3.26(-20) |
| $\|\|u - u_{M-N}\|\|_{0,\partial S}$ | 3.38(-2) | 2.81(-4) | 1.50(-7) | 1.00(-10) | 7.44(-14) | 5.85(-17) | 4.78(-20) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{\infty,\partial S}$ | 1.14(-1) | 1.77(-3) | 1.72(-6) | 1.69(-9) | 1.63(-12) | 1.56(-15) | 1.59(-18) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{0,\partial S}$ | 1.72(-1) | 2.71(-3) | 2.69(-6) | 2.65(-9) | 2.60(-12) | 2.54(-15) | 2.49(-18) |
| $\sigma_{\max}$ | 5.75 | 8.58 | 1.19(1) | 1.44(1) | 1.66(1) | 1.85(1) | 2.03(1) |
| $\sigma_{\min}$ | 2.25(-1) | 6.54(-1) | 7.38(-3) | 9.69(-4) | 1.35(-4) | 1.95(-5) | 2.88(-6) |
| Cond | 2.55(1) | 1.31(2) | 1.61(3) | 1.49(4) | 1.23(5) | 9.49(5) | 7.05(6) |
| Cond_eff | 8.73 | 4.40(1) | 5.38(2) | 4.98(3) | 4.10(4) | 3.17(5) | 2.35(6) |

At last, an alternative method is also implemented which is called the Truncation Singular Value Decomposition (TSVD). The numerical results by the TSVD are listed in Table 8. We can see almost the same accuracy and good stability for those obtained from the over-determined system Equations 29 and 30. In particular, the modified condition numbers in Table 8 display the nearly linear growth for $(\epsilon, \bar \epsilon) = (0,0)$ as,

$$\text{Cond}^*(\mathbf{A}) = O(M), \quad \text{Cond\_eff}^*(\mathbf{A}) = O(M),$$

to restore a good stability. Also by the TSVD, for $(\epsilon, \bar \epsilon) = (0.5,0.2)$, Table 9 provides almost the same rates as those given in Table 7 by the over-determined system. However, if carefully comparing the results from these two techniques, the errors and condition numbers from the over-determined system are slightly better than the TSVD. As a result, we will recommend the engineers to use the simple overdetermined system to overcome the instability of the psudo-singularity of the collocation method from the conservative NFM.

Table 8: Errors and condition numbers for Model Problem by the conservative schemes with $M = 2N$ and $(\epsilon, \bar{\epsilon}) = (0, 0)$ by the TSVD.

| $(M,N)$ | (4, 2) | (10, 5) | (20, 10) | (30, 15) | (40, 20) | (50, 25) | (60, 30) |
|---|---|---|---|---|---|---|---|
| $\|\|u - u_{M-N}\|\|_{\infty,S}$ | 6.36(-2) | 5.54(-4) | 2.95(-7) | 1.96(-10) | 1.44(-13) | 1.12(-16) | 9.27(-20) |
| $\|\|u - u_{M-N}\|\|_{0,\partial S}$ | 1.00(-1) | 8.08(-4) | 4.18(-7) | 2.76(-10) | 2.03(-13) | 1.59(-16) | 1.30(-19) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{\infty,\partial S}$ | 3.51(-1) | 5.87(-3) | 5.94(-6) | 5.87(-9) | 5.75(-12) | 5.60(-15) | 5.57(-18) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{0,\partial S}$ | 4.90(-1) | 8.20(-3) | 8.21(-6) | 8.10(-9) | 7.94(-12) | 7.78(-15) | 7.61(-18) |
| $\sigma_{\max}$ | 5.53 | 8.66 | 1.22(1) | 1.49(1) | 1.72(1) | 1.92(1) | 2.11(1) |
| $\sigma_{\min}$ | 1.58(-3) | 9.77(-6) | 4.23(-9) | 2.40(-12) | 1.52(-15) | 1.03(-18) | 7.25(-22) |
| $\sigma_{min-next}$ | 4.35(-1) | 2.75(-1) | 1.76(-1) | 1.38(-1) | 1.18(-1) | 1.04(-1) | 9.44(-2) |
| Cond | 3.50(3) | 8.86(5) | 2.88(9) | 6.22(12) | 1.13(16) | 1.87(19) | 2.91(22) |
| Cond_eff | 1.13(3) | 2.80(5) | 9.15(8) | 1.98(12) | 3.59(15) | 5.94(18) | 9.26(21) |
| Cond* | 1.27(1) | 3.14(1) | 6.94(1) | 1.08(2) | 1.46(2) | 1.85(2) | 2.23(2) |
| Cond_eff* | 4.17 | 9.95 | 2.21(1) | 3.43(1) | 4.66(1) | 5.88(1) | 7.11(1) |

Table 9. Errors and condition numbers for Model problem by the conservative schemes with M=2N and $(\epsilon, \bar{\epsilon}) = (0.5, 0.2)$ by the TSVD.

| $(M,N)$ | (4, 2) | (10, 5) | (20, 10) | (30, 15) | (40, 20) | (50, 25) | (60, 30) |
|---|---|---|---|---|---|---|---|
| $\|\|u - u_{M-N}\|\|_{\infty,S}$ | 6.48(-2) | 5.49(-4) | 2.85(-7) | 1.87(-10) | 1.36(-13) | 1.06(-16) | 8.76(-20) |
| $\|\|u - u_{M-N}\|\|_{0,\partial S}$ | 1.04(-1) | 8.30(-4) | 4.21(-7) | 2.75(-10) | 2.01(-13) | 1.57(-16) | 1.28(-19) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{\infty,\partial S}$ | 3.54(-1) | 5.84(-3) | 5.77(-6) | 5.63(-9) | 5.48(-12) | 5.32(-15) | 5.27(-18) |
| $\|\|u_\nu - (\hat{u}_{M-N})_\nu\|\|_{0,\partial S}$ | 5.01(-1) | 8.42(-3) | 8.32(-6) | 8.11(-9) | 7.91(-12) | 7.72(-15) | 7.53(-18) |
| $\sigma_{\max}$ | 5.26 | 8.26 | 1.16(1) | 1.43(1) | 1.65(1) | 1.84(1) | 2.01(1) |
| $\sigma_{\min}$ | 3.07(-4) | 2.12(-7) | 2.40(-12) | 3.55(-17) | 5.88(-22) | 1.04(-26) | 1.90(-31) |
| $\sigma_{min-next}$ | 3.16(-1) | 6.54(-2) | 7.38(-3) | 9.69(-4) | 1.35(-4) | 1.95(-5) | 2.88(-6) |
| Cond | 1.71(4) | 3.89(7) | 4.85(12) | 4.01(17) | 2.80(22) | 1.77(27) | 1.06(32) |
| Cond_eff | 5.81(3) | 1.29(7) | 1.61(12) | 1.34(17) | 9.32(21) | 5.91(26) | 3.53(31) |
| Cond* | 1.67(1) | 1.26(2) | 1.58(3) | 1.47(4) | 1.22(5) | 9.42(5) | 7.00(6) |
| Cond_eff* | 5.72 | 4.19(1) | 5.25(2) | 4.90(3) | 4.05(4) | 3.14(5) | 2.33(6) |

(2) Interior Field Method

In Theorem A of section 2.2, when $u \in H^2(\partial S) \cup u_\nu \in H^1(\partial S)$, the NFM (Equations 15-16) works for field nodes $Q \in \partial S$, that is, $\rho = R$ and $\bar{\rho} = R_1$ on $\partial S_R$ and $\partial S_{R_1}$, respectively. So for Equations 15-16 should still hold on the condition of $\rho = R$ and $\bar{\rho} = R_1$ on $\partial S_R$ and $\partial S_{R_1}$. The solution of the field nodes inside domain should be able to obtain by Equation 21, i.e.,

$$u_{M-N}=u_{M-N}(\rho,\theta)=u_{M-N}(\bar{\rho},\bar{\theta})=a_0-R(\ln R)p_0-R_1(\ln R_1)\bar{p}_0$$
$$+\frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho^k}{R^k}\right)(p_k\cos(k\theta)+q_k\sin(k\theta))+\frac{R}{2}\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)(a_k\cos(k\theta)+b_k\sin(k\theta)) \quad (31)$$
$$+\frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\bar{\rho}}\right)^k(\bar{p}_k\cos(k\bar{\theta})+\bar{q}_k\sin(k\bar{\theta}))+\frac{R_1}{2}\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\bar{\rho}^k}\right)(\bar{a}_k\cos(k\bar{\theta})+\bar{b}_k\sin(k\bar{\theta}))$$

And those unknown coefficients $[a_i,b_i,p_i,q_i,\bar{a}_i,\bar{b}_i,\bar{p}_i,\bar{q}_i]^T$ can be obtained by suitable collocation scheme. The main idea is that when Equation 31 with $\rho=R$, then we can have the following equations:

$$u_{M-N}(\rho,\theta,\bar{\rho},\bar{\theta})=f(\theta)=a_0+\sum_{k=1}^{M}\{a_k\cos k\theta+b_k\sin k\theta\}, \text{ on } \partial S_R \quad (32)$$

and similarly, when $\bar{\rho}=R_1$, we have,

$$u_{M-N}(\rho,\theta,\bar{\rho},\bar{\theta})=g(\theta)=\bar{a}_0+\sum_{k=1}^{M}\{\bar{a}_k\cos k\bar{\theta}+\bar{b}_k\sin k\bar{\theta}\}, \quad \text{on } \partial S_{R_1} \quad (33)$$

Equations 32-33 can be used to solve the unknown coefficients, without having to solve Equations 15-16 first and then substitute into Equation 21 to find the solutions for field nodes inside domain. This is the new Interior Field Method, we name this method because we use directly the first equation of the Green formula in Equation 9, that the field nodes are located inside domain S.

The following numerical experiment and analysis also have been done in the research.

1. Can we rewrite the Interior Field Method (that is, Equations 32-33) by the conservative scheme (after some manipulation of Equation 26)?

**The answer is yes.**

After we have obtained the results of the variables, we can also derive the interior solution, which is a conservative interior solution

$$u_{M-N}^{Conser} = u_{M-N}^{Conser}(\rho,\theta) = u_{M-N}^{Conser}(\overline{\rho},\overline{\theta}) = a_0 - R(\ln\frac{R}{\overline{\rho}})p_0$$

$$+\frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho^k}{R^k}\right)(p_k\cos(k\theta)+q_k\sin(k\theta))+\frac{R}{2}\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)(a_k\cos(k\theta)+b_k\sin(k\theta)) \qquad (33\text{-}1)$$

$$+\frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k(\overline{p}_k\cos(k\overline{\theta})+\overline{q}_k\sin(k\overline{\theta}))+\frac{R_1}{2}\sum_{k=1}^{N}\left(\frac{R_1^{\,k-1}}{\overline{\rho}^k}\right)(\overline{a}_k\cos(k\overline{\theta})+\overline{b}_k\sin(k\overline{\theta}))$$

Equation 33-1 can be used to derive the Interior field method and their conservative scheme further. In the next section, we like to compare some boundary methods, such as the Collocation Trefftz method, Null field method, and Interior field method, etc.

(3) Collocation Trefftz Method

A collocation Trefftz method can be implemented to compare numerical solutions with NFM, Conservative scheme NFM, IFM, and Conservative Interior Field Method.

By [Li,2009], the particular solutions of Collocation Trefftz Method can be given as:

$$\begin{aligned}&u_{M-N}(\rho,\theta,\overline{\rho},\overline{\theta}) = a_0 + \sum_{i=1}^{M}\left(\frac{\rho}{R}\right)^i(a_i\cos i\theta + b_i\sin i\theta) + \\ &\overline{a}_0\ln\overline{\rho} + \sum_{i=1}^{N}\left(\frac{R_1}{\overline{\rho}}\right)^i(\overline{a}_i\cos i\overline{\theta}+\overline{b}_i\sin i\overline{\theta})\end{aligned} \quad ;\ \rho \le R,\ \overline{\rho}\ge R_1 \qquad (34)$$

where $(a_i, b_i, \overline{a}_i, \overline{b}_i)^T$ are unknown coefficients to be determined.

We can examine if Equation 31 (the interior field method) and Equation 34 are similar, so we can say that IFM is a special case of CTM. Some relations between these coefficients between IFM and CTM have been carried out in the project to reassure the finding that IFM is a special case of CTM.  We have drawn a figure to demonstrate the relationship between several boundary methods, such as NFM, IFM, CTM, and BIE, and is given as the following Figure 3..

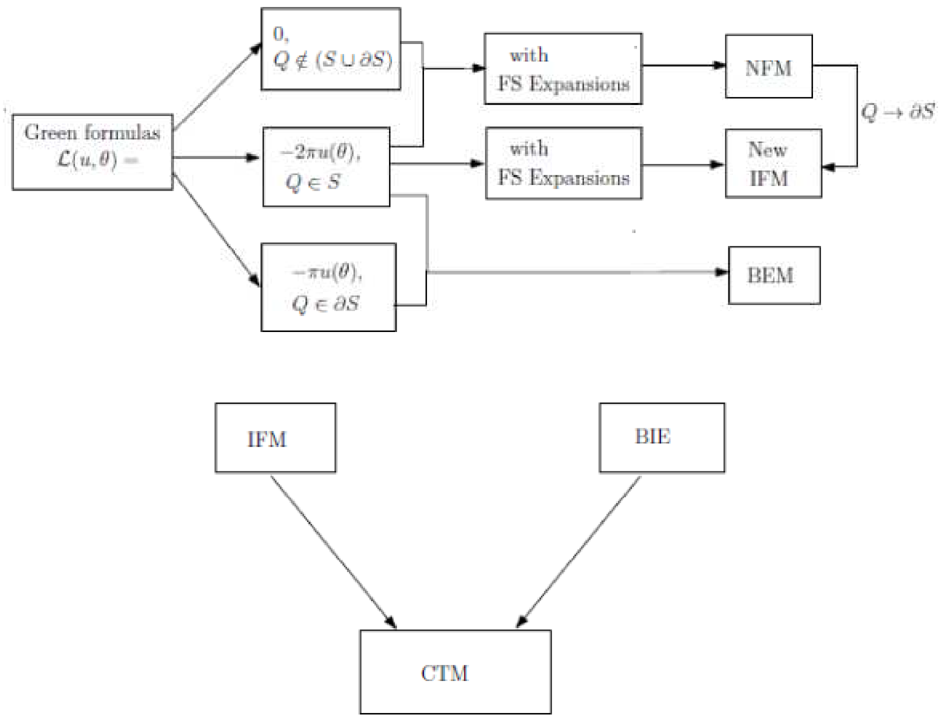


Figure3 : Relationship between NFM, IFM, CTM and BIE

(4) Some error analysis

Error analysis of the case for eccentric circular domain is also studied in the project. Some Sobolev norms for Fourier functions are provided in [Canuto, Quarteroni, 1982]. The analysis is difficult as it always should be, we have studied and give the result given as the following, and the proof is skip here.

**Theorem** A: For the concentric circular domain when $R.\neq. 1$, the leading coefficients are exact by the NFM, and the solution errors are resulted only from the truncations of their Fourier expansions.

The convergence of the numerical result by the NFM is given by the Theorem B and is proved by the following two lemmas.

**Lemma 4.1** *Let (4.1) be given, for $\partial S_R = \ell_R$, there exist the bounds of the remainders of (4.7) and (4.8)*

$$\|u - \bar{u}^M\|_{q,\partial S_R} \le C \frac{1}{M^{p-q}} |u|_{p,\partial S_R}, \quad 0 \le q \le p, \tag{4.11}$$

$$\|u_\rho - \bar{u}_\rho^M\|_{q,\partial S_R} \le C \frac{1}{M^{p-q-1}} |u_\rho|_{p-1,\partial S_R}, \quad 0 \le q \le p-1, \tag{4.12}$$

*where $C$ is a constant independent of $M$.*

**Lemma 4.2** *Let (4.1) be given, for $\partial S_{R_1} = \ell_{R_1}$, there exist the bounds of the remainders of (4.13) and (4.14),*

$$\|u - \bar{u}^N\|_{q,\partial S_{R_1}} \leq C\frac{1}{N^{p-q}}|u|_{p,\partial S_{R_1}}, \quad 0 \leq q \leq p, \tag{4.15}$$

$$\|u_{\bar{\nu}} - \bar{u}^N_{\bar{\nu}}\|_{q,\partial S_{R_1}} \leq C\frac{1}{N^{p-q-1}}|u_{\bar{\nu}}|_{p-1,\partial S_{R_1}}, \quad 0 \leq q \leq p-1, \tag{4.16}$$

*where $C$ is a constant independent of $N$.*

**Theorem 4.1** *Let (4.1)* **and** *$R \neq 1$ hold. For the solution $u_{N,M}$ from the TM in (2.40), there exists the error bound,*

$$\|u - u_{N,M}\|_{0,\partial S} \leq C\{\frac{1}{M^p}|u|_{p,\partial S_R} + \frac{1}{N^\sigma}|u|_{\sigma,\partial S_{R_1}}\}, \tag{4.17}$$

*where $C$ is a constant independent of $N$ and $M$.*

(5) Numerical Experiments for IFM and its Conservative scheme.

For Equations 1-3, the Model problem, we choose $\varepsilon = \bar{\varepsilon} = 0$, and use the IFM in Equation 31, and the conservative scheme in Equation 26 with symmetry, the explicit equation for the interior field solution is rewritten as:

$$\begin{aligned} u_{M-N} &= u_{M-N}(\rho,\theta) = u_{M-N}(\bar{\rho},\bar{\theta}) = a_0 - R(\ln R)p_0 - R_1(\ln R_1)\bar{p}_0 \\ &+ \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho^k}{R^k}\right)p_k\cos(k\theta) + \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\bar{\rho}}\right)^k\bar{p}_k\cos(k\bar{\theta}) \end{aligned} \tag{35-a}$$

Also the conservative IFM solution can be used and is given as:

$$\begin{aligned} u^{Conser}_{M-N} &= u_{M-N}(\rho,\theta) = u_{M-N}(\bar{\rho},\bar{\theta}) = a_0 - R(\ln\frac{R}{R_1})p_0 \\ &+ \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho^k}{R^k}\right)p_k\cos(k\theta) + \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\bar{\rho}}\right)^k\bar{p}_k\cos(k\bar{\theta}) \end{aligned} \tag{36-a}$$

The computation is similar to Equation 32-33, but with $\varepsilon = \bar{\varepsilon} = 0$.

Also, two conservative IFM solutions can be obtained as the following:

$$L^{Conserv}_{ext}(R,\theta,R_1,\bar{\theta}) := -R(\ln\frac{R}{R_1})p_0 + \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}p_k\cos(k\theta) + \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\bar{\rho}}\right)^k\bar{p}_k\cos(k\bar{\theta}) = 0,$$

where $(r,\theta)$ on $\partial S_R$ (37-a)

$$L^{Conserv}_{\text{int}}(R,\theta,R_1,\bar{\theta}) := -2\pi R(\ln\frac{R}{R_1})p_0 + R\pi\sum_{k=1}^{M}\frac{1}{k}(\frac{\rho}{R})^k p_k\cos(k\theta) + R_1\sum_{k=1}^{N}\frac{1}{k}\left(\frac{\bar{\rho}}{R_1}\right)^k\bar{p}_k\cos(k\bar{\theta}) = 0,$$

where $(r,\bar{\theta})$ on $\partial S_{R_1}$ (37-b)

An algorithm like the Conservative NFM Scheme is designed but with $\varepsilon=\bar{\varepsilon}=0$. That is a new system with 2(M+N)+1 equations, of 2(M+N)+1 field nodes,

Conservative IFM scheme:

$$w_i L_{ext}^{Conserv}(R,i\Delta\theta,\bar{\rho}_i,\bar{\theta}_i)=0,\ \ i=0,1,2,...,2M \tag{38-a}$$

$$w_i L_{\text{int}}^{Conserv}(\rho_i,\theta,R_1,i\Delta\bar{\theta})=0,\ \ i=0,1,2,...,2N-1 \tag{39-a}$$

Afterward, some comparisons between the IFM and the NFM have been made, both with and without conservative scheme.

Based on the work in [Li, Huang, Liaw, Lee*, 2012], we have the similar theorems.

**Theorem B:** Let $u\in H^{p}(\partial S)\cup u_{\nu}\in H^{p-1}(\partial S)(p\geq 2)$ be given. The Interior Field Method is just a special case of NFM with $\varepsilon=\bar{\varepsilon}=0$ (in Equations 17-18 or 27-30).

**Theorem C:** Let S be a simple annular domain with symmetric solution to x-axis. When $R\neq 1$, for the algebraic equations $F^{*}x=b^{*}$ obtained by Equation 17-18 or Equation 27-28 or Equation 29-30, there exists the bound

$$Cond(F^{*})=\frac{\lambda_{\max}(F^{*})}{\lambda_{\min}(F^{*})}=O(N) \tag{40-a}$$

The importance of this project is to let us understand more on the performance of the conventional Null Field Method and to examine its variation such as the IFM and their conservative schemes, and study their numerical performance, and compare these new methods by the criteria of stability and convergence rates. Hope the results can help engineers know better of these methods and to implement in their applications properly.

**結果與討論：這個計畫的執行產生幾篇 SCI 論文的發表，**

(一)、『The Null Field Method of Dirichlet Problems of Laplace's equation on Circular domain with circular holes』

這是一篇在2012年三月發表在 Engineering Analysis with Boundary Elements [SCI, IF 1.531-1.704 5 years]的文章，作者含李子才教授、黃宏財教授、及學生廖采頻，我是通訊作者。這篇文章主要是針對海洋大學陳正宗特聘教授及其研究團隊發展的一個邊界積分方程的數值方法稱為零場法做明顯解的探討。該方法是將基本解以退化核方式將基本解用在邊界元素法產生之格林公式內。已知或未知的圓形邊界值以富立葉級數展開，利用正交性產生代數方程式。針對零場法在邊界布點的合理性在這篇文章證明是可行的施行方式，而且對於病態性的情形作了理論證明，我們也發現在邊界或是場域裡布點不會影響太多準確性。穩定性是放在邊界上最好，雖然這有違背零場的意義，但是這篇文章已經證明了可以布點在邊界上了。若是在邊界外布點則離邊界越遠穩定性越差。這篇文章的很多計算經驗都可以用到接下來的計畫中。

(二)

Lee, Ming-Gong, Li, Z.C., Huang, H.T.*, Chiang, John*, 『*Conservative schemes and degenerate scale problems in the null-field method for Dirichlet problems of Laplace's equation in circular domains with circular holes,*』 Engineering Analysis with Boundary Elements, (2013)Vol. 37(1), pp. 95-106.[**SCI] (IF:1.451-1.531-5years, Engineering**, 19/90; mathematics applications, 22/92, 0 cite**). 發表在今年(2013)一月份的國際 SCI 期刊** Engineering Analysis with Boundary Elements。

這篇論文的內容是先討論守恆的條件之下對於零場法的修正結果，也很意外的發現這個守恆的演算法對於在計算上的一種退化尺度問題竟然可以完全避開，所以在計算可以給工程師一個很重要的參考，當然這與原來的零場法還是提供一個改進的方法。

**其他的論文發表：**

(三) 延續去年度的研究計畫的線彈性體的結果，在 2013 年三月在國際頂尖 SCI 期刊 Numerical Methods for Partial Differential Equations 發表。『The Collocation Trefftz Methods for Stokes Equations with Singularity, Numerical Methods for Partial Differential Equations』, Vol. 29, Issue 2, pp. 361–395, March 2013. DOI: 10.1002/num.21710.[SCIE:IF:1.404:1.374-5 years, mathematics applied, 33/245), 0 cite] 這論文使用主持人在線彈性體計畫中所找到得特別解在 Stokes 問題的計算上使用。以下參考文獻包含主持人在這一兩年的新著作及正在進行中的工作，主持人一直持續在這個領域做出貢獻。

**References:**

[Abramowitz et al. 1964] Abramowitz, M. and Stegun, I.A., Handbook of Mathematical Functions with

Formulas, Graphs and Mathematical Tables, Dover Publications, Inc., New York, 1964.
[Bulatov, **Lee**, 2010] Bulatov M.V., **Lee, Ming-Gong**, Solovarova, L.S. (2010, 9), *On First- and Second-Oredr Difference Schemes for Differential-Algebraic Equations of Index at Most Two*, Computational Mathematics and Mathematical Physics, Sep, 2010, Vol. 50, No. 11, pp. 1808-1817. (English version) **[SCI] (IF: 0.03)**
[Bulatov, **Lee**, 2008] Bulatov, M.V. and **Lee, Ming-Gong**, (2008, 12), *Application of Matrix Polynomials to the Analysis of Linear Differential-Algebraic Equations of Higher Order*, Differential Equations, Vol. 44, No. 10, pp. 1299-1305. **(SCI-Expanded, I.F. 0.35)**
[Canuto, Quarteroni, 1982] Canuto C. and Quarteroni, A., Approximation results for orthogonal polynomials in Sobolev spaces, Math. Comp., Vol. 38, pp67-86, 1982.
[Chen, Zhou, 1992] G. Chen and J. Zhou, Boundary Element Methods, Academic Press, Chapter. 9, New York, 1992.
[Chen, Liao, Lee, 2009] Chen, J.T., Liao, H.Z., and Lee, W.M., An analytical approach for the Green's functions of biharmonic problems with circular and annular domains, Journal of Mechanics, Vol. 25, No. 1, pp. 59-74, March 2009.
[Chen, Lee, Lee, 2009] Chen J.T., Lee, Y.T., and Lee, W.M., A semi-analytical approach for boundary value problems with circular boundaries, Recent Advances in Boundary Elements Methods, G.D. Manolis, and D. polyzos (eds.), December, 2009.
[Chen, Shen, Wu, 2005] J. T. Chen, W. C. Shen and A. C. Wu, "Null-field integral equations for stress field around circular holes under anti-plane shear." Eng Anal Bound Elem 30 (2005), 205–217.
[Chen et al. 2008a] J. T. Chen, C. T. Chen and I. L. Chen, "Null-field integral equation approach for eigenproblems with circular boundaries." J. Comp. Acoustics 15 (2008), 1–28.
[Chen et al. 2008b] J. T. Chen, P. Y. Chen and C. T. Chen, "Surface motion of multiple alluvial valleys for incident plane SH-waves by using a semi-analytical approach." Soil Dyn Earthqu Eng 28 (2008), 58–72.
[Chen et al. 2008c] J. T. Chen, C. T. Chen, P. Y. Chen and I. L. Chen, "A semi-analytical approach for radiation and scattering problems with circular boundaries." Comput Methods Appl Mech Eng 196 (2008), 2751–2764.
[Chen, Kuo, Lin, 2002] J. T. Chen, S. R. Kuo and J. H. Lin, Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity, Engineering Analysis with Boundary Elements, Vol. 26, pp. 1669 – 1681, 2002.
[Chen, Lee, Chen, Lin, 2002] J. T. Chen, C. F. Lee. J. L. Chen and J. H. Lin, An alternative method for degenerate scale problem in boundary element methods for two-dimensional Laplace equation, Engineering Analysis with Boundary Elements, Vol. 26, pp. 559 – 569, 2002.
[Chen, Lee, Liao, 2008] J. T. Chen, W. M. Lee and H. Z. Liao, "Isotropic clamped-free thin annular circular plate subjected to a concentrated load." ASME J Appl Mech 75 (2008), 658–663
[Chen, Shen, 2009] J. T. Chen, and W. C. Shen, Null-field approach for Laplace problems with circular boundaries using degenerate kernels, Numer. Meth. PDE., vol. 25, pp. 63 – 86, 2009.
[Huang, Li, **Lee**, Chiang, 2013] Huang, Hung-tsai, Li, Zi-Cai, Lee, Ming-Gong, and Chiang, John Y., *New Algorithm and Error Analysis of the Interior Field Method for Laplace's Equation on Circular Domain with Circular Holes, submitted to Journal of Computaional and Applied Mathematics, March 18, 2013* [SCI] (IF:1.292)
[Huang, **Lee***, Li, Chiang, 2013]Huang, Hung-tsai, Lee, Ming-Gong*, Li, Zi-Cai, and Chiang, John Y., *Null*

*field and interior field methods for Laplace's equation in actually punctured disks,* accepted by the special issue "Generalized Differential and Integral Equations" of Abstract and Applied Analysis, September 2013 [SCIE] (IF:1.318, mathematics, 18/289, Mathematics applied, 43/245)

[**Lee**, Young, Chu, Li, 2011] *Mixed Types of Boundary Conditions at Corners of Linear Elastostatics and Their Numerical Solutions*, Engineering Analysis with Boundary Elements, *Volume 35, Issue 12, December 2011, Pages 1265-1278.*

[**Lee**, Young, Li, Chu, 2010] *Combined Trefftz Methods of Particular and Fundamental Solutions for Corner and Crack Singularity of Linear Elastostaatics*, Engineering Analysis with Boundary Elements, Vol. 34, pp 632~654, 2010.

[**Lee**, Li, Zhang, Huang, Chiang, 2013] Lee, Ming-Gong, Li, Zi-Cai, Zhang, Liping, Huang, Hung-Tsai, and Chiang, John Y., *Algorithm Singularity of the Null-Field Method for Dirichlet Problems of Laplace's Equation in Annular an Circular Domains, Under revision by Engineering Analysis with Boundary Elements, April 8, 2013.* [SCI] (IF:1.451-1.531-5years, Engineering, 19/90; mathematics applications, 22/92).

[Li, Chu, Young, **Lee***, 2010] Li, Z. C. , Chu, P.C. (學生), Young L. J., and **Lee, Ming-Gong** , *Models of corner and crack singularity of linear elastostatics and their numerical solutions*, Engineering Analysis with Boundary Elements, Vol. 34, pp. 533~548, 2010.

[Li, **Lee**, Chiang, 2013] Li, Z.C., Lee, Ming-Gong, and Chiang* John, *Error Analysis of the Method of Fundamental Solutions for Linear Elastostatics*, Journal of Computational and Applied Mathematics, Vol. 251, pp.133-153, **Oct. 2013**.**[SCI] (IF:1.292)**

[Li, Young, **Lee***, Tsai, 2011] Li, Z.C., Young, L.J., **Lee, Ming-Gong*** and Tsai, W.C. (學生), *Hybrid and Collocation Trefftz Methods for Traction Boundary Conditions in Linear Elastostatics*, Engineering Analysis with Boundary Elements, Vol. 35, pp. 944-958, Oct., 2011. [**SCI] (IF:1.531-1.704-5 years)**

**[**Li, **Lee***, Chen, 2011**]** Li, Z.C., **Lee, Ming-Gong***, and Chen, J.T., *New Series Expansions for Fundamental Solutions of Linear Elastostatics in 2D*, Computing, Vol. 92, No. 3, June 2011, pp. 199~224.

[Li, **Lee**, Chiang*, 2011] Li, Z.C., **Lee, Ming-Gong**, Chiang* John, and Liu, Ya-Ping, *The Trefftz Method using Fundamental Solutions for Biharmonic Equations*, Journal of Computational and Applied Mathematics, Vol. 235, Issue 15, June 1, 2011, pp. 4350-4367.

[Li, 1998] Z.C. Li, Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities, Kluwer Academic Publishers, Boston (1998).

[Li, 2009] Z.C. Li, Method of fundamental solutions for annular shaped domains, J. Comp. and Appl. Math., vol. 228, pp. 355 - 372, 2009.

[Li, Mathon, 1987] Z.C. Li, R. Mathon and P. Sermer, Boundary methods for solving elliptic problems with singularities and interfaces, SIAM J. Numer. Anal., Vol. 24(3), pp.487-498, 1987.

[Li, Huang, Huang, 2010] Z. C. Li, J. Huang and H. T. Huang (2010), Stability analysis of method of fundamental solutions for mixed boundary value problems of Laplace's equation, Computing vol. 88, pp. 1 – 29, 2010. 2010.

[Li, Lu, 2000] Z.C. Li and T.T. Lu, Singularities and treatments of elliptic boundary value problems, Mathematical and Computer Modeling, vol. 31, pp. 79-145, 2000.

[Li, Lu, Huang, Cheng, 2007] Z. C. Li, T. T. Lu, H. T. Huang and A. H. D. Cheng, Trefftz, collocation and other coupling methods - A comparison, Numer. Meth for PDEs, Vol 23, pp. 93-144, 2007.

[Li, Lu, Hu, Cheng, 2008] Z. C. Li, T. T. Lu, H. Y. Hu and A. H. D. Cheng, Trefftz and Collocation Methods, WIT press, Southampton, Boston, January 2008.

[Li, Huang, Liaw, **Lee***, 2011] Z.C. Li, H. T. Huang, C. P. Liaw, and M. G. Lee, The null-field method of

Dirichlet problems of Laplace's equation on circular domains with circular holes, 36(2012), pp. 477-491, Enginnering Analysis with Bundary Elements, March 2012.
[Li, Lee, Chiang*, 2012] Li, Z.C., Lee, Ming-Gong, and Chiang* John, The Collocation Trefftz Methods for Stokes Equations with Singularity, Numerical Methods for Partial Differential Equations, Vol. 29, Issue 2, pp. 361–395, March 2013. DOI: 10.1002/num.21710.[SCIE:IF:1.404:1.374-5 years, mathematics applied, 33/245), 0 cite]
[Li, Lee, Chiang*, Liu, 2011] Li, Z.C., Lee, Ming-Gong, Chiang* John, and Liu, Ya-Ping, The Trefftz Method using Fundamental Solutions for Biharmonic Equations, Journal of Computational and Applied Mathematics, Vol. 235, Issue 15, June 1, 2011, pp. 4350-4367. [SCI] (IF:1.292)
[Li, Huang, Wei, 2011] Z. C. Li, H. T. Huang and Y. Wei, ill-conditioning of the truncation singular value decomposition and the Tikhonov regularization and their application to numerical partial differential equations, Numerical Linear Algebra with Applications, Vol. 18, pp. 205-221, 2011.
[Palaniappan, 2002] D. Palaniappan, Electrostatoics of two intersecting conducting cylinders, Math. Comput Modelling, Vol. 36, pp. 821 – 830, 2002.
[Yu, 2002] D. H. Yu, Natural Boundary Integral Method and its Applications, Science Press/Kluwer Academic Publishers, Beijing/New York, 2002.

**已完成之工作項目及成果**

在 99 年度研究計畫中所規劃的線彈性體的計算，很多的結果已經發表在國際重要期刊 Engineering Analysis with Boundary Elements **[SCI, IF :1.531-1.704-5 years]**，包含[Li, Tsai, Lee*, Young 2010] 、[Lee, Young, Li, Chu, 2011]；目前還有一篇文章『*Corners and Crack Singularity of Mixed Types Boundary Conditions for Linear Elastostatics and Their Numerical Solutions*』也已經發表。所以上年度的計畫許多相關結果都已經在國際期刊發表。

這兩年陸續已經有多篇文章發表在 SCI 期刊上(請參閱前頁目錄)。 **其中與今年度的研究計畫相關的一篇文章[Li, Huang, Liaw, Lee*, 2012]已經在 2012 年三月發表**，許多在本計畫提出的新想法都是墊基於這篇文章所作出的延伸，包含穩定性及某些誤差分析的工作，而這些理論的依據就是這篇文章。與這篇文章不同處，在於我們將推導出與原來方法不相同的表現型態，及另一種符合物理性質 Flux 守恆的型態，包括 Dirichlet 及 Neumann 等類型及其混合型的總體檢討。所以我們在這篇論文『The null-field method of Dirichlet problems of Laplace's equation on circular domains with circular holes』的計算上已經累積部分心得，尤其在零場法的顯示推導及部分誤差分析等。在這兩年的計畫之中，我們將對零場法的演變型態做出貢獻。在 2014 年 Trefftz 的重要國際會議"The Joint conference of The sixth Trefftz and the second FMS"會在墨西哥繼續舉行，Eng. Anal. Bound. Elem.期刊的編輯，國際研究人員及國際上研究相關主題學者將會與會，期在這之前能就零場法的研究有重要的結果累積，此研究案的成果將與

國內外研究人員套討論並繼續改進。

茲將以上的研究主題，及各個階段已經完成的子題依研究性質及相關技術成果以表格方式展現，工作表列如下：

| **Project Title:**<br>**在邊界積分方程的新內場法研究** | | |
|---|---|---|
| **A Conservative Scheme of Null Field Method and Interior Field Method**<br>**for Laplace's Equation on Dirichlet Boundary Condition** | | |
| 項目/研究主題/研究性質 | | 相關參考資料 |
| A conservative scheme for Null Field Method<br>And Its Explicit algebraic expressions | (理論) | 1. Chen, Shen, Wu, 2005<br>2. Chen et al. 2008a,b,c<br>3. Chen, Shen 2009 |
| Interior Field Method<br>-without Conservative scheme<br>-with Conservative scheme | (理論) | 1.Li, Huang, Liaw, Lee, 2012 |
| Collocation Trefftz Method | (理論)<br>(程式技術) | 1. Li, 2009 |
| Error analysis of the case for eccentric circular domain will be studied in the project. The Sobolev norms for Fourier functions are needed. | (理論) | 1. Canuto and Quarteroni, 1982<br>2. Li, Lu, Hu,Cheng, 2008 |
| Numerical Experiments for IFM and its Conservative scheme | (程式技術) | 1. Li, Huang, Liaw, Lee, 2012 |

# 行政院國家科學委員會補助國內專家學者出席國際學術會議報告

101 年 9月 20 日

| | | | |
|---|---|---|---|
| 報告人姓名 | 李明恭 | 系所<br>職稱 | 休閒遊憩規劃與管理學系 |
| 時間<br>會議<br>地點 | September 10~ September 14, 2012<br>Vienna, Austria<br>維也納 奧地利 | 核定字號 | 101-2914-I-216-002-A1 |
| 會議<br>名稱 | (中文)<br>(英文) 6th EUROPEAN CONGRESS ON COMPUTATIONAL METHODS IN APPLIED SCIENCES AND ENGINEERING (ECCOMAS 2012) | | |
| 發表<br>論文<br>題目 | (中文)<br>(英文) **The Interior Field Method for Laplace's equation on circular domains with circular holes** | | |

一、考察參觀活動(無是項活動者省略)
二、建議
三、攜回資料名稱及內容
四、其他

**報告內容應包括下列各項：**

**五、 參加會議經過**

此次會議是國際重要的應用科學的計算方法(computational method in applied sciences)研討會，本次參加人數約2100人，超過六十一個國家的數學家、工程師、科學家參加，由歐洲(European Community on Computational Methods in Applied Sciences ECCOMAS 主辦，並由Institute for Mechanics of Materials and Structures and the Institute of Lightweight Design and Structural Biomechanics of the Vienna University of Technology 協辦，在奧地利維也納大學(維也納)舉行五天。台灣此次有幾位參加，但是由於會場太大遂無法一一認出，只知道一位國立海洋大學河海工程學系范佳銘教授、台大應力所Prof. Chen, C.D. 擔任semi-plenary speech。我主要行程是參加一個mini-symposium『Recent advances in boundary element and meshless methods』, 由 Prof. Zhang C. and Prof. Sladek V. 兩位教授主辦召集而形成，這個國際會議共有108 個mini-symposium同時舉行，而這個mini-symposium是當中最多人參加的，總共有31篇論文報告，時間從星期二到星期四，其報告論文的題目包含邊界元的方法及計算；及邊界積分方程利用collocation Trefftz method 計算的各種問題，非常廣泛，收穫良多。我的報告是位在星期四最後一場的第二位報告，我的題目涵蓋最近我們(作者包含中山大學李子才教授(已於今年七月在中山大學退休)、義守大學黃宏財教授、及中山大學蔣依吾教授四位)在基本的拉普拉斯方程、Dirichlet problems使用零場法的計算方法的研究。零場法是海洋大

學終身講座教授陳正宗教授於近年提出在解決邊界積分方程的一個方法，陳教授也在去年獲得教育部工程學門學術獎章。零場法的使用哲學是必須將source點放置在場域之外，但是事實上使用者皆將source點放在場域的邊界上而形成計算實施方面場點配置的矛盾。所以我們覺得有將此方法繼續研究的必要，第一是必須對此使用方法的矛盾解是其合法性並作理論證明。我們在今年年初這份論文已經在國際SCI期刊EABE(Engineering Analysis with Boundary Elements)發表，在這篇論文中另一個重要的結果是將source點的放置地點作一個位置上的定性解釋，我們的結論就是如果source點放至於邊界上那麼穩定性是最好，但是準確性稍差，收斂性沒有因為位置而有太大的改變，所以最好的位置是離邊界不要太遠的位置來放置source點，這個位置與環型場域的圓半徑有一些比例性的關係，輔合這個比例的位置可以給計算時得到比較好的穩定性及準確性，而且還有指數收斂的快速結果，這個結果可以給工程師使用相似方法時一個重要的參考依據，詳細結果請參閱論文『The null-field method of Dirichlet problems of Laplace’ s equation on circular domains with circular holes』, Engineering Analysis with Boundary Elements 36 (2012) 477– 491.

另外我們也發展出有別於目前零場法的一個新的計算方法，我們在新的計算方法的想法是根據Flux 守恆的要求，還根據零場法的推導發現一個比較省事的方法目前稱為內場法，是由場域內的解直接將其推展至外圓及內圓的邊界而得，與目前零場法推導的方法來比較需要四組方程才能得到代數方程，內廠法只需一組方程即可，當然理論證明當source點推至於邊界時這兩種方法是等價的，這也是目前工程人員使用的零場法方式。另一個重要的發現是守恆型的零場法

可以巧妙的避開所謂因為研究場域的幾何形狀而產生計算時不穩定或是沒有唯一解的奇異現象，我們稱為degenerate scale problems，這在計算數學或是工程計算時非常重要，但是因為報告時間只有二十分鐘，所以我無法談論太多這方面的議題。會後本次mini-symposium的主辦人Prof. Zhang 也非常同意我的觀點並對這個問題有高度的興趣，並鼓勵我繼續在這個問題繼續進行探究。

我在我們的minisymposium結束之後當天下午又參加另一個mini-symposium 『Method of Fundamental Solutions 』，是由美國學者Prof. C.S. Chen所主辦，但是由於某些關係他沒有親自參加，但是因為參加的許多報告學者皆有參加去年在台灣高雄中山大學主辦的Joint Conference of Trefftz & Fundamental Solutions 國際研討會。由於我們的研究與基本解也非常有關所以特別去聽。其中海洋大學的范佳銘老師講的問題Stokes 問題，因為與我們去年的研究問題相關所以特別注意，然而我發現他的結論有一些值得繼續注意的小問題，范老師也說在兩年前陳正宗教授聽他演講時就提出相同的疑問，我告知這與我之前在微分代數方程的研究有些相關，心裡當然有些高興因為我與陳教授提出相同的問題，希望這個問題能繼續與范教授聯繫解決。

在會場與國際出版社人員談到出版的事情，他非常希望我們能夠提供出版的書，看來不管台灣或是國際上其他學者出版新的研究相關的書籍的意願越來越低了，期望能有機會將這幾年的結果出版研究專書。

**六、　攜回資料名稱及內容**

一本大會的論文摘要集 另外我也要了一張海報非常漂亮，是奧地利畫家Belvedere所畫的一張名畫稱為『吻(the kiss)』。

無研發成果推廣資料

# 101 年度專題研究計畫研究成果彙整表

計畫主持人：李明恭　　計畫編號：101-2115-M-216-001-

計畫名稱：在邊界積分方程的新內場法研究

| 成果項目 | | | 量化：實際已達成數（被接受或已發表） | 量化：預期總達成數(含實際已達成數) | 量化：本計畫實際貢獻百分比 | 單位 | 備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等） |
|---|---|---|---|---|---|---|---|
| 國內 | 論文著作 | 期刊論文 | 0 | 0 | 100% | 篇 | |
| | | 研究報告/技術報告 | 0 | 0 | 100% | | |
| | | 研討會論文 | 0 | 0 | 100% | | |
| | | 專書 | 0 | 0 | 100% | | |
| | 專利 | 申請中件數 | 0 | 0 | 100% | 件 | |
| | | 已獲得件數 | 0 | 0 | 100% | | |
| | 技術移轉 | 件數 | 0 | 0 | 100% | 件 | |
| | | 權利金 | 0 | 0 | 100% | 千元 | |
| | 參與計畫人力（本國籍） | 碩士生 | 0 | 0 | 100% | 人次 | |
| | | 博士生 | 0 | 0 | 100% | | |
| | | 博士後研究員 | 0 | 0 | 100% | | |
| | | 專任助理 | 0 | 0 | 100% | | |
| 國外 | 論文著作 | 期刊論文 | 2 | 0 | 100% | 篇 | 一篇在國際期刊 Engineering Analysis with Boundary Elements (SCI)發表；另一篇已經被 Abstract and Applied Analysis (SCIE)接受 |
| | | 研究報告/技術報告 | 0 | 0 | 100% | | |
| | | 研討會論文 | 3 | 0 | 100% | | 在 ECCOMAS２０１２、2012 海峽兩岸破壞科學與材料實驗研討會、及 Taiwan SIAM meeting, BIEM/BEM symposium 發表 |
| | | 專書 | 0 | 0 | 100% | 章/本 | |
| | 專利 | 申請中件數 | 0 | 0 | 100% | 件 | |
| | | 已獲得件數 | 0 | 0 | 100% | | |
| | 技術移轉 | 件數 | 0 | 0 | 100% | 件 | |
| | | 權利金 | 0 | 0 | 100% | 千元 | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 參與計畫人力（外國籍） | 碩士生 | 2 | 0 | 100% | 人次 | |
| | | 博士生 | 0 | 0 | 100% | | |
| | | 博士後研究員 | 0 | 0 | 100% | | |
| | | 專任助理 | 0 | 0 | 100% | | |
| 其他成果<br>(無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。) | | 無 | | | | | |

| | 成果項目 | 量化 | 名稱或內容性質簡述 |
|---|---|---|---|
| 科教處計畫加填項目 | 測驗工具(含質性與量性) | 0 | |
| | 課程/模組 | 0 | |
| | 電腦及網路系統或工具 | 0 | |
| | 教材 | 0 | |
| | 舉辦之活動/競賽 | 0 | |
| | 研討會/工作坊 | 0 | |
| | 電子報、網站 | 0 | |
| | 計畫成果推廣之參與（閱聽）人數 | 0 | |

# 國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以 100 字為限）

      □實驗失敗

      □因故實驗中斷

      □其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：■已發表 □未發表之文稿 □撰寫中 □無

   專利：□已獲得 □申請中 ■無

   技轉：□已技轉 □洽談中 ■無

   其他：（以 100 字為限）

   Lee, Ming-Gong, Li, Z.C., Huang, H.T.*, Chiang, John*, Conservative schemes and degenerate scale problems in the null-field method for Dirichlet problems of Laplace's equation in circular domains with circular holes, Engineering Analysis with Boundary Elements, (2013)Vol. 37(1), pp. 95-106.[SCI] (IF:1.451-1.531-5 years, Engineering, 19/90； mathematics applications, 22/92, 0 cite).

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

   在工程上很多解決的演算法通常只看數值結果，並與文獻作比較而已，但是在我們的論文中了解數值姐的收斂性及穩定的性質是非常重要的，所以在計畫中也對於此工程上很重要的演算法提供更精確的理論依據。另外也針對此一傳統的數值方法一個新的演進，一個新的方法在這個計畫中產生，並依據通量守恆的原則設計一個守恆的模式，對於舊方法的改變是非常重要的演伸，也因此發現這個新的手恆演算可以避功成是一個很重要的陷阱-degenerate scale problems- 是因為問題場域幾何形狀所引起的問題，雖然也因此產生另一個條件系數變大的情形，但是我們有找了另一個解決的途徑。總而言之，這個計畫的實施再計算數學方面提供一個比目前更多的一個計算選項。