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圖的邊數、退化數與貝蒂差值

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摘要

圖的退化數$\varsigma(G)$(圖的貝蒂差值$\xi(G)$)定義為連通圖 G 去除其生成樹的邊後所留下之最小連通子圖之個數(奇數邊的連通子圖的個數)。而在過去的幾年中，我們一直致力於給予圖的直徑與連通性的圖，其最大虧格(或貝蒂差值)與退化數之研究。而在此問題之研究中，我們發現了直徑與連通性同時為n的圖，其中$n=2$或 3，其邊數、退化數與貝蒂差值有著密切的關係。在此之前，Murty 曾正明直徑與連通性同時為 2 的圖，邊數大於等於二倍的邊數減 5，而 Skoviera 則證明了直徑與連通性同時為 2 的圖，其貝蒂值與退化數同時小於等於 4。後來，我們更證明此兩定理具有等價關係，甚至我們亦發現上式等式成立之極端圖若排除可能的自我迴路(loops)外，則此圖均相同。另外，在直徑與連通性同時為 3 的圖，我們亦發現其邊數、退化數與貝蒂差值亦具有相同的特性，因此本計畫主要在於研究圖的邊數、退化數與貝蒂差值之相關性，在此計畫中，我們找出了具有相同退化數與貝蒂差值圖的特性。而具有相同退化數與貝蒂差值圖的最小邊數為我們未來之研究方向。

關鍵詞：邊數、退化數、貝蒂差值

Edge Number, Decay Number and Betti Deficiency of Graphs

Ming-Chun Tsai

Abstract

Let the decay number of G, $\varsigma(G)$ (resp. the Betti deficiency, $\xi(G)$) be the minimum number of components (resp. odd size components) of a co-tree of a connected graph G. In the past years, we study the maximum genus (or Betti deficiency) and decay number of graphs with given diameter and connectivity. In the study of this problem, we find that for $n=2$ or 3, if k is the minimum number such that $q \geq 2p-k$ for each $(p,q)-$graph with connectivity n and diameter n, and l is the minimum number such that $\xi(G) \leq \varsigma(G) \leq l-1$ for each graph G with connectivity n and diameter n, then $k=l$. On the other hand, Murty and Skoviera find that $k=5$ and $l=5$ for $n=2$, respectively. Furthermore, we defined three classes of extremal graphs in the above inequalities and we proved that they are the same, with the exception of loops added to vertices. This intrigues us to study the relation among the edge number, decay number and Betti deficiency of graphs. In this project, we find some properties for the graph G with $\xi(G)=\zeta(G)$. In the further, the lower bound of edge number for the graph G with $\xi(G)=\zeta(G)$ is the direction of our research.

Keywords: edge number, decay number, Betti deficiency

1. Preliminary

Throughout this report, a graph may have multiple edges or loops, but a simple graph contains neither multiple edges nor loops. For *G* connected, let *T* be a spanning tree of *G*. Denote by $\xi(G-T)$ the number of components with an odd number of edges of the co-tree $G-T$. Clearly, $\xi(G)$ is the minimum value of $\xi(G-T)$ over all co-trees of *G*. The invariant $\xi(G)$ was introduced in [8] to calculate the maximum genus $\gamma_M(G)$ of G by the formula $\gamma_M(G)=(\beta(G)-\xi(G))/2$, where $\beta(G)$ is the Betti number. In [9], Skoviera defined the decay number of G, $\varsigma(G)$ to be the minimum number of components of a co-tree of a connected graph G. It is clear that $\xi(G)\le\varsigma(G)$ for any graph. For *G* connected, Nebesky[6,7] discovered formulas to calculate $\xi(G)$ and $\varsigma(G)$.

Theorem 1.1(Nebesky[6]) *For any connected graph* G*,*

$$\xi(G)=\max\{c(G-A)+b(G-A)-|A|-1 \mid A\subseteq E(G)\},$$

where $c(G-A)$ *and* $b(G-A)$ *denote the number of components and the number of odd Betti number components in* $G-A$*.*

Theorem 1.2(Nebesky [7]) *For any connected graph* G*,*

$$\varsigma(G)=\max\{2c(G-A)-|A|-1 \mid A\subseteq E(G)\}.$$

In [9], Skoviera gave a tight upper bound on $\xi(G)$ and $\varsigma(G)$ for 2-connected graph G of diameter 2.

Theorem 1.3(Skoviera[5]) *If* G *is a* 2*-connected, diameter* 2 *graph, then*

$$\xi(G)\le\varsigma(G)\le 4.$$

This theorem, together with Theorem 1.2, gives another proof of the following theorem which was discovered by Murty[5].

Theorem 1.4(Murty[5]) *If* G *is a* 2*-connected, diameter* 2 $(p,q)-$*graph, then*

$$q\ge 2p-5.$$

In the above theorems, we find that there are some relations among edge number, decay number and maximum genus of graphs. Motivated by this results, we study the edge number, decay number and maximum genus of graphs.

In [2], Fu, Tsai and Xuong defined three classes of extremal graphs in

Theorem 1.1 and 1.3 and proved that they are the same, with the exception of loops added to vertices. For 3-connected, diameter 3 graphs, Tsai and Fu proved in [11] the following theorems.

Theorem 1.5(Tsai and Fu[11]) *Let* Ω *be the collection of all* 3*-connected diameter* 3 *graphs. If* k *is the minimum number such that* $q \geq 2p - k$ *for each* $(p,q)-$*graph* $G \in \Omega$*, and* l *is the minimum number such that* $\varsigma(H) \leq l - 1$ *for each graph* $H \in \Omega$*, then* $k = l$*.*

In [11], we also proved the following theorem.

Theorem 1.6(Tsai and Fu[6]) *If* G *is a* 3*-connected,* $(p,q)-$*graph of diameter 3, then*

$$q \geq 2p - 11.$$

By Theorem 1.5 and 1.6, we have that $k = l \leq 11$ and then $\xi(G) \leq \varsigma(G) \leq 10$ for any *3-connected diameter 3 graph* G.

2. The Main Results

Let G be a connected graph, if T is a spanning tree of G with $\varsigma(G) = c(G - T)$, then T is called a mc-tree (minimum number of components tree) of G. On the other hand, if T is a spanning tree of G with $\xi(G) = \xi(G - T)$, then T is called a splitting tree of G.

Theorem 2.1. *For any mc-tree* T *of* G*,* $\xi(G - T) = c(G - T)$ *if and only if* $\xi(G) = \zeta(G)$

Proof. Let $\xi(G - T) = c(G - T)$ for any mc-tree T of G. Assume that $\xi(G) < \zeta(G)$. There exists a splitting tree T' of G which is not a mc-tree of G. Then there exist $e_1 \in E(T'), e_1' \in E(G - T)$ such that $T'' = T' \setminus \{e_1\} \cup \{e_1'\}$ with $c(G - T'') < c(G - T')$ and $\xi(G - T'') = \xi(G - T')$. If T'' is not a mc-tree of G, then we can use the same process to find a mc-tree T^* of G such that

$$c(G - T^*) < \cdots < c(G - T'') < c(G - T') \quad \text{and} \quad \xi(G - T^*) = \cdots = \xi(G - T'') = \xi(G - T')$$

$$\xi(G) = \xi(G - T') = \xi(G - T^*) = c(G - T^*) = \varsigma(G).$$

This is a contradiction.

Conversely, let $\xi(G) = \zeta(G)$. If T is a mc-tree of G, then

$$c(G - T) = \zeta(G) = \xi(G) \leq \xi(G - T),$$

This implies that $\xi(G - T) = c(G - T)$. □

Corollary 2.2. *Let G admits a connected cotree, then* $\zeta(G)=\xi(G)$ *if and only if* $\beta(G)$ *is even.*

In [4], Kundu proved that every 4-connected graph contains two edge disjoint spanning tree, hence by Corollary 2.2 ,we have the following corollary.

Corollary 2.3. *If G is a 4-connected graph, then* $\zeta(G)=\xi(G)$ *if and only if* $\beta(G)$ *is even.*

Theorem 2.6. *If* $\xi(G)=\zeta(G)$*, then* $G-T$ *has no isolated vertex for any splitting tree* T *of* G.

Proof. Assume that $\xi(G)=\zeta(G)$. Let T be a spanning tree of G such that $\xi(G)=\xi(G-T)=c(G-T)=\zeta(G)$. Since $G-T$ has at least one isolated vertex,

$$\xi(G)=\xi(G-T)<c(G-T)=\zeta(G)$$

This is a contradiction. So we have the proof. □

Corollary 2.7. *If* $\xi(G)=\zeta(G)$*, then G has no cut vertex.*

As Figure 1, for the graph *G*

$$\xi(G)\le\xi(G_1)+\xi(G_2)+\cdots+\xi(G_m)<\varsigma(G_1)+\varsigma(G_2)+\cdots+\varsigma(G_m)+1=c(G-T)$$

for any spanning tee T. Thus $\xi(G)<\zeta(G)$ for the graph *G*.

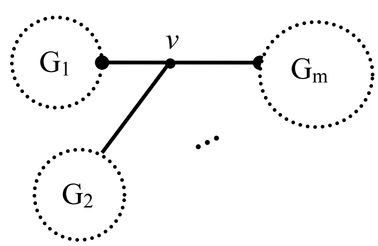

Figure 1

3. **Concluding Remark**

In this report, we find some properties for the graph *G* with $\xi(G)=\zeta(G)$. For the graph *G* with $\xi(G)=\zeta(G)$, we can get its Betti number by finding the decay number. In the further, the lower bound of edge number for the graph *G* with $\xi(G)=\zeta(G)$ is the direction of our research.

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